Chapter 4 Integration

Student Notes

By: Cole ℝidgway

Contents

4.1 Antiderivatives and Indefinite Integration

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Write the general solution of a differential equation.	I can write the general solution of a differential equation, including the constant of integration.
Use indefinite integral notation for antiderivatives.	I can correctly use indefinite integral notation to represent families of antiderivatives.
Use basic integration rules to find antiderivatives.	I can apply basic integration rules to find antiderivatives of elementary functions.
Find a particular solution of a differential equation.	I can find a particular solution of a differential equation by using given initial conditions.

4.1.1 Antiderivatives

Definition of an Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note: F is called *an* antiderivative, rather than *the* antiderivative. We represent the entire family of antiderivatives by adding a constant C.

Theorem 4.1: Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form:

$$G(x) = F(x) + C$$

for all x in I where C is a constant.

Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $\dfrac{d}{dx}[x^2] = 2x$ you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C$$

where C is a constant. The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f. and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.

Example 4.1: Find the general solution to the differential equation $y' = 2$.

4.1.2 Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)dx$$

The operation of finding all solution of this equation is called **antidifferentiation** or **indefinite integration** and is denoted by the integral sign $\int$. The general solution is denoted by

$$y = \int f(x)dx = F(x) + C$$

The expression $\int f(x)dx$ is read as the *indefinite integral of f with respect to x*. So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

4.1.3 Basic Integration Rules

Inverse relationship between Differentiation and Integration:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

Basic Integration Rules

Differentiation Formula	**Integration Formula**
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

Example 4.2: Describe the antiderivative of $3x$.

Strategy: Rewrite functions into familiar forms (specifically power rule x^n) before integrating.

Example 4.3: Integrate the following:

1. $\displaystyle\int \frac{1}{x^3}\,dx$

2. $\displaystyle\int \sqrt{x}\,dx$

Example 4.4: Integrate the polynomial: $\int (3x^4 - 5x^2 + x)\, dx$

Strategy: Split fractions or use trig identities to simplify before integrating.

Example 4.5:

$$\int \frac{x+1}{\sqrt{x}} dx$$

Example 4.6:

$$\int \frac{\sin(x)}{\cos^2(x)} dx$$

Practice Exercises

Evaluate the following indefinite integrals

1. $\displaystyle\int (3x^4 - 5x^2 + 2)\, dx$

2. $\displaystyle\int \left(\sqrt[3]{x} + \frac{1}{x^3}\right)\, dx$

3. $\displaystyle\int \frac{x^2 + 2x - 3}{\sqrt{x}}\, dx$

4. $\displaystyle\int (x + 2)(2x - 3)\, dx$

5. $\displaystyle\int (4\sin(x) - 3\csc^2(x))\, dx$

6. $\displaystyle\int \frac{\cos(t)}{\sin^2(t)}\, dt$

4.1.4 Initial Conditions and Particular Solutions

To find a **particular solution**, we must determine the specific value of C using an **Initial Condition** (a known point (x, y)).

Example 4.7: Find the particular solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

that satisfies the initial condition $F(1) = 0$.

Example 4.8: A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

(a) Find the position function giving the height s as a function of time t. ($a(t) = -32$)

(b) When does the ball hit the ground?

Practice Exercises

Solve the following initial value and motion problems

7. Find the particular solution $y = f(x)$ that satisfies the differential equation $f'(x) = 6x^2 - 4x + 2$ and the initial condition $f(1) = 9$.

8. Find the function $y(x)$ if $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and $y(9) = 4$.

9. Find the particular solution for $f(x)$ given the second derivative $f''(x) = 6x$, with initial conditions $f'(0) = 2$ and $f(0) = 5$.

10. A particle moves along the x-axis with a velocity given by $v(t) = 3t^2 - 2t$ for $t \geq 0$. If the particle is at position $x = 3$ when $t = 1$, find the position of the particle at $t = 3$.

11. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 64 feet per second. Use the acceleration due to gravity $a(t) = -32$ ft/sec^2.

1. Find the position function $s(t)$.

2. Determine the maximum height reached by the ball.

12. An object is dropped from a cliff that is 400 feet high. Its acceleration is $a(t) = -32$ ft/sec^2 and initial velocity is $v(0) = 0$. Determine the velocity of the object at the moment it impacts the ground.

4.2 Area

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Use sigma notation to write and evaluate a sum.	I can write a sum using sigma notation and evaluate it correctly.
Understand the concept of area.	I can explain area as an accumulation of infinitely many small regions.
Approximate the area of a plane region.	I can approximate the area of a plane region using rectangles and sums.
Find the area of a plane region using limits.	I can find the exact area of a plane region by taking the limit of a Riemann sum.

4.2.1 Sigma Notation

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation**, a_i is the i**th** term of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 4.9: Expand or evaluate the following sums:

1. $\displaystyle\sum_{i=1}^{6} i$

2. $\displaystyle\sum_{i=0}^{5} (i+1)$

3. $\sum_{j=3}^{7} j^2$

4. $\sum_{k=1}^{n} \frac{1}{n}(k^2+1)$

5. $\sum_{i=1}^{n} f(x_i)\Delta x$

Properties of Summation

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Theorem 4.2: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 4.10: Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$.

In the table, note that the sum appears to approach a limit as n increases. Although the discussion of limits at infinity in Section 3.5 applies to a variable x, where x can be any real number, many of the same results hold true for limits involving the variable n, where n is restricted to positive integer values. So, we can find the limit

$$\lim_{n\to\infty} \frac{n+3}{2n} = \frac{1}{2}$$

Practice Exercises

Evaluate the sums and limits

13. Evaluate the sum: $\sum_{k=1}^{5}(2k-3)$

14. Use the summation formulas to evaluate: $\sum_{i=1}^{20}(2i^2-3)$

15. Write the sum in sigma notation: $\frac{2}{n}\left(1+\frac{2}{n}\right)^2+\frac{2}{n}\left(1+\frac{4}{n}\right)^2+\cdots+\frac{2}{n}\left(1+\frac{2n}{n}\right)^2$

16. Simplify the summation formula for n terms: $\sum_{i=1}^{n}\frac{3i}{n^2}$

17. Find the limit of the sum as $n\to\infty$: $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n^2}+\frac{5}{n}\right)$

18. Given that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, evaluate $\lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$.

4.2.2 Area

Euclidean geometry provides simple formulas for polygons. We can determine the area of any polygon by subdividing it into triangular regions.

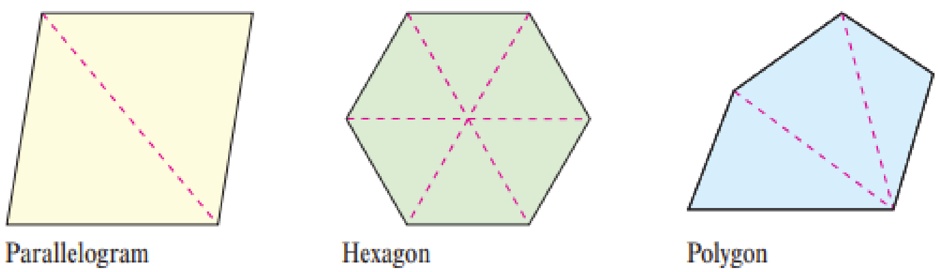


Figure 4.1

However, finding the area of curved regions is more difficult. The ancients used the **Exhaustion Method**, squeezing the area between inscribed and circumscribed polygons.

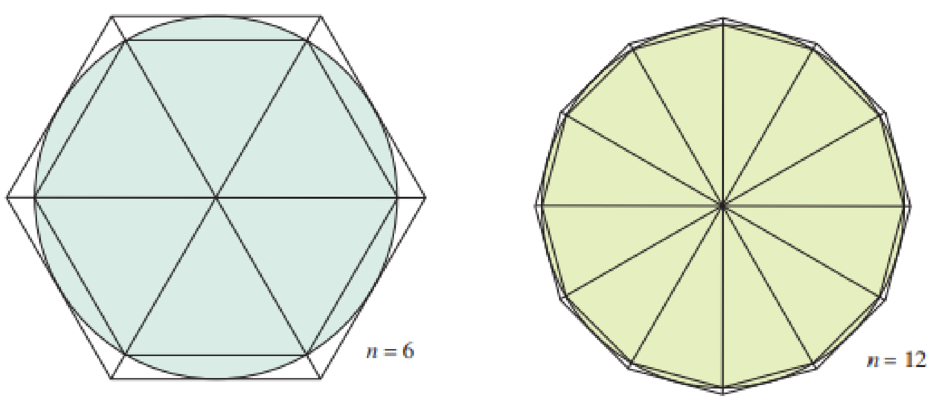


Figure 4.2 The exhaustion method for finding the area of a circular region.

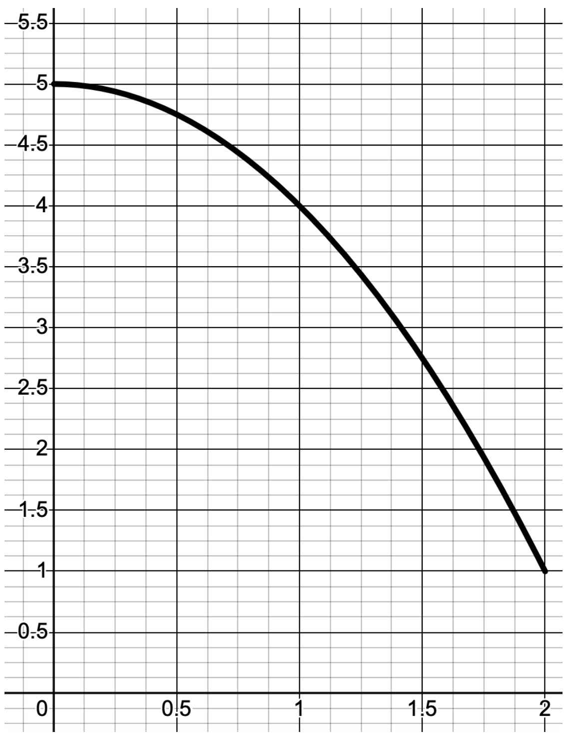
4.2.3 The Area of a Plane Region

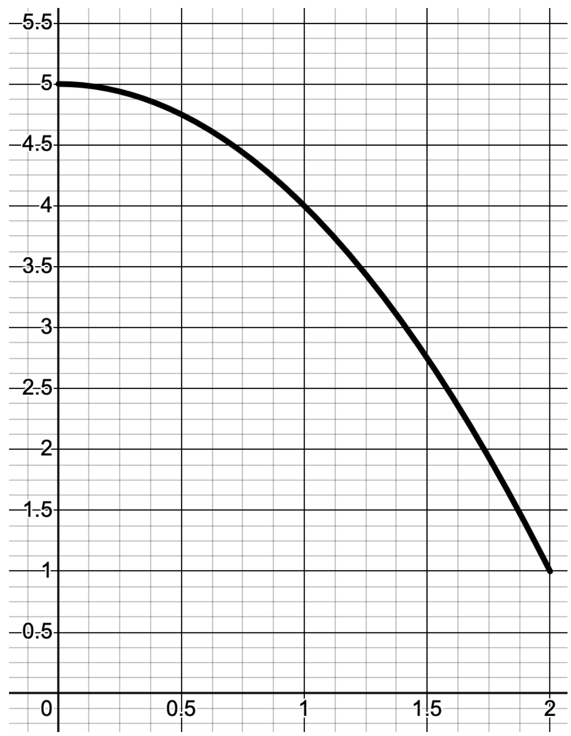
We will use a similar process to Archimedes, but using rectangles (Riemann Sums).

Example 4.11: Use five rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.





4.2.4 Upper and Lower Sums

The procedure in the previous example can be generalizes as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$. The region is bounded below by the x-axis, and the left and right boundaries of the region are vertical lines $x = a$ and $x = b$.

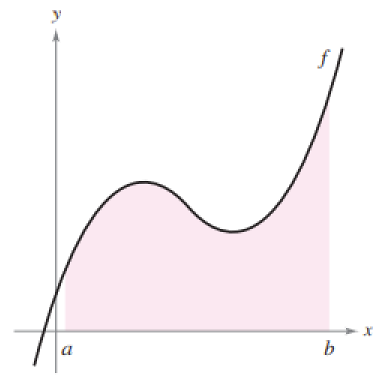


Figure 4.3 The region under a curve

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$.
The endpoints of each intervals are as follows:

$$\underbrace{a + 0(\Delta)x}_{a=x_0} < \underbrace{a + 1(\Delta)x}_{x_1} < \underbrace{a + 2(\Delta)x}_{x_2} < \cdots < \underbrace{a + n(\Delta)x}_{x_n=b}$$

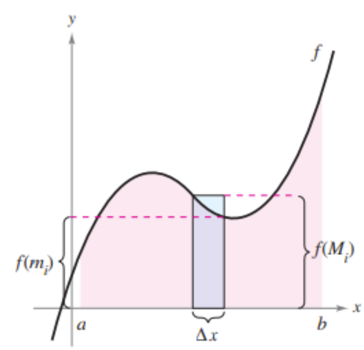


Figure 4.4 Interval $[a, b]$ divided into n subintervals

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum anda maximum value of $f(x)$ in *each* subinterval.

$$\begin{aligned} f(m_i) &= \text{Minimum value of } f(x) \text{ in } i\text{th subinterval} \\ f(M_i) &= \text{Maximum value of } f(x) \text{ in } i\text{th subinterval} \end{aligned}$$

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \leq f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of circumscribed rectangles is called an **upper sum**.

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$$
$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$$

In the figure below, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between the two sums.

$$s(n) \leq (\text{Area of Region}) \leq S(n)$$

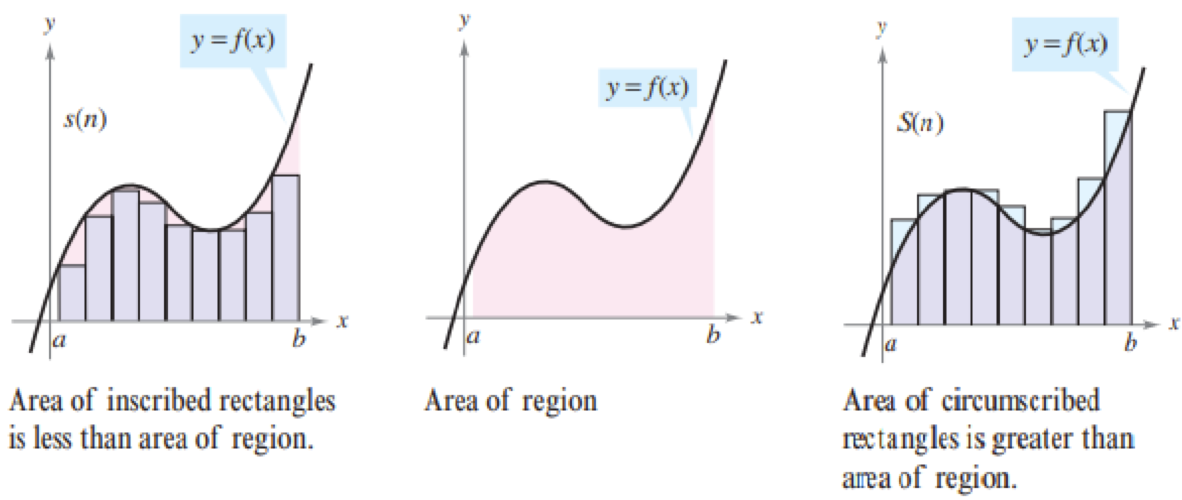


Figure 4.5

Example 4.12: Find the upper and lower sums for the region bounded by the graph $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.
Setup: Partition the interval $[0, 2]$ into n subintervals, each of width $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.
To begin, partition the interval $[0, 2]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Because f is increasing on the interval $[0, 2]$, the minimum value of each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Lower Sum calculation ($s(n)$):

Upper Sum calculation ($S(n)$):

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n. the lower sum is less than (or equal to) the upper sum.

$$s(n) == \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limits as $n \to \infty$, both the upper sum and lower sum converge to $\frac{8}{3}$

Theorem 4.3: Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem that the choice of x in the ith subinterval does not affect the limit. This means that the you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane.*

Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$

Example 4.13: Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$ using the limit definition.

4.3 Riemann Sums and Definite Integrals

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Understand the definition of a Riemann sum.	I can explain what a Riemann sum represents and how it is constructed from partitions and sample points.
Evaluate a definite integral using limits.	I can evaluate a definite integral by writing it as a limit of Riemann sums.
Evaluate a definite integral using properties of definite integrals.	I can use properties of definite integrals to simplify and evaluate integrals without using limits.

4.3.1 Riemann Sum

I would like to note that I vary somewhat heavily from the textbook in this section. While I believe that the textbook does a really good job *mathematically*, I think my approach is more intuitive.
With that said, I maintain the same definitions and theorems presented in the book, if you choose to reference them.

The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In the preceding section, the funciton f was assumed to be continuous and nonnegative because we were dealing with area under a curve).

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

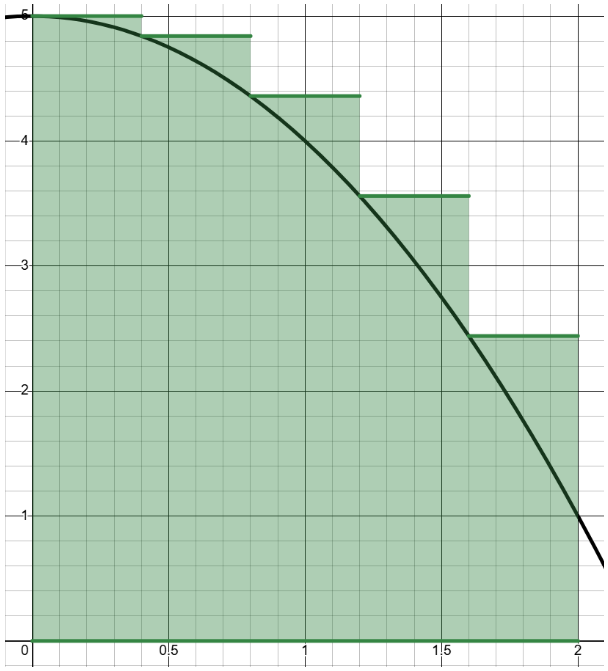
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

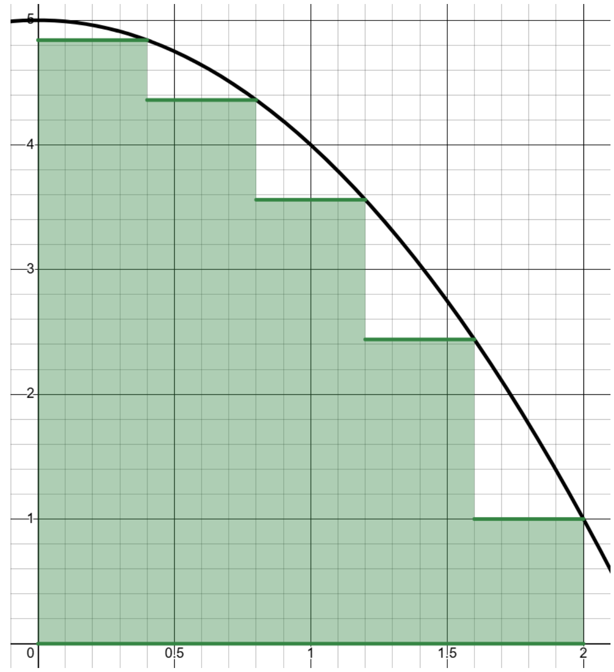
where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

We previously explored Riemann Sums in 4.2.3. The example was $f(x) = -x^2 + 5$





In this example, we explored **Left-End Riemann Sums** and **Right-End Riemann Rums** to find that their sums were

$$\sum_{i=1}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08$$

and

$$\sum_{i=1}^{5} f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48$$

Example 4.14: Calculate the left and right Riemann sum of the function $f(x) = x^2 + x$ using 5 subintervals on the interval $[0, 3]$.

Step 1: Determine width of subintervals (Δx)

Step 2: List the partition points (x_i)

Step 3: Calculate the Left Sum (L) *Using endpoints:* x_0, x_1, x_2, x_3, x_4

Step 4: Calculate the Right Sum (R) *Using endpoints:* x_1, x_2, x_3, x_4, x_5

Notice that the key difference between the left and the right Riemann sums comes from the selection of the endpoints.
You can visualize the differences easily using desmos.

Practice Exercises

19. Let $f(x) = x^2 + 2$. Use a right Riemann sum with $n = 4$ equal subintervals to approximate the area of the region bounded by the graph of f and the x-axis on the interval $[0, 2]$.

20. The function f is continuous and strictly increasing on the interval $[1, 5]$. A table of selected values is given below.

x	1	2	3	4	5
$f(x)$	2	4	7	11	18

Using 4 subintervals of equal length, determine if the Left Riemann Sum is an overestimate or an underestimate of the actual area under the curve. Justify your answer.

21. Use a Midpoint Riemann sum with $n = 3$ equal subintervals to approximate the area under the curve $f(x) = \frac{1}{x}$ on the interval $[1, 7]$.

22. The function $f(x) = \ln(x)$ is continuous on the interval $[2, 10]$.

1. Write the summation notation for the right Riemann sum approximation of the area under the graph of $f(x)$ on $[2, 10]$ using n subintervals of equal width.

2. Calculate the approximation using $n = 4$ subintervals.

23. Consider the region bounded by $f(x) = \sqrt{x}$ and the x-axis on the interval $[0, 9]$. Set up, but do not evaluate, the expression for the Right Riemann sum using n equal subintervals.

4.3.2 Definite Integrals

Partitions and Norms

- The width of the largest subinterval of a partition Δ is the **norm**, denoted by $||\Delta||$.
- **Regular Partition:** All subintervals have equal width.

$$||\Delta|| = \Delta x = \frac{b-a}{n}$$

- **General Partition:** Subintervals may vary in width.

$$||\Delta|| \to 0 \implies n \to \infty$$

Definition of a Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

exists, then f is **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

The limit is called the **definite integral** of f from a to b.

- a is the **lower limit** of integration.
- b is the **upper limit** of integration.

Important Distinction:

Definite Integral	Indefinite Integral
$\int_a^b f(x)\,dx$	$\int f(x)\,dx$
Result is a **Number**	Result is a **Family of Functions** $(+C)$

A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

A sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this class.

Theorem 4.4: Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

Theorem 4.5: The Definite Integral as the Area of a Region

If f is continuous and **nonnegative** on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)dx$$

In strict geometric terms, "area" is always positive. However, in Calculus, we use **Net Signed Area**.

Theorem: The Definite Integral as Net Signed Area

If f is continuous on the closed interval $[a, b]$, the definite integral represents the **net signed area** bounded by the graph of f and the x-axis.
If we let A_{up} be the area of the geometric region between the graph of f and the x-axis where $f(x) \geq 0$, and A_{down} be the area of the geometric region where $f(x) < 0$ (as shown in Figure 4.6), then:

$$\int_a^b f(x)\, dx = A_{up} - A_{down}$$

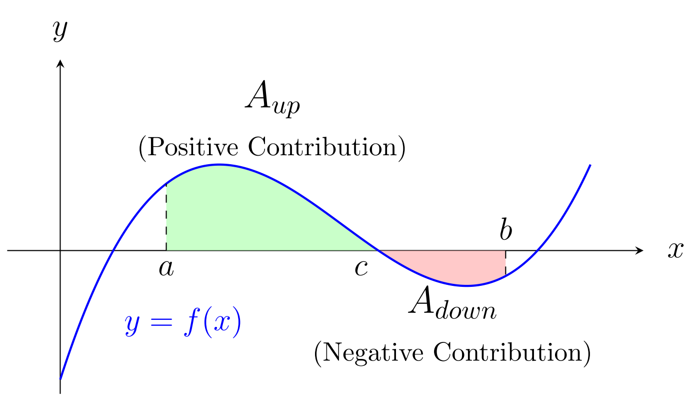


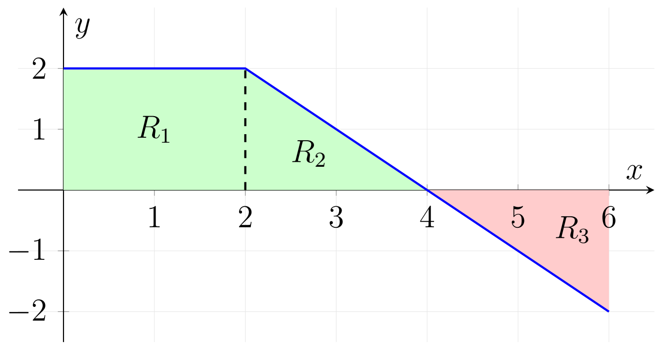
Figure 4.6 Visualizing Net Signed Area

Summary of Area Concepts:

1. **Net Signed Area (Displacement):**
   - The value of the definite integral.
   - Regions below the x-axis are negative.
   - Net Area $= \int_a^b f(x)\,dx = A_{up} - A_{down}$

2. **Total Geometric Area (Distance):**
   - The sum of the absolute areas (all treated as positive).
   - Calculated by integrating the *absolute value* of the function.
   - Total Area $= \int_a^b |f(x)|\,dx = A_{up} + A_{down}$

Example 4.15: Consider the function f defined by the graph below. Evaluate the definite integral $\int_0^6 f(x)\,dx$ and calculate the total area of the region between the graph and the x-axis.



Begin by identifying the geometric shapes formed by the graph and the x-axis.

- Region R_1 (on $[0,2]$) is a rectangle with width 2 and height 2.
- Region R_2 (on $[2,4]$) is a triangle with base 2 and height 2.
- Region R_3 (on $[4,6]$) is a triangle with base 2 and height 2 (below the axis).

First, calculate the geometric area of each shape:

$$\begin{aligned}
\text{Area}(R_1) &= \text{width} \cdot \text{height} = 2 \cdot 2 = 4 \\
\text{Area}(R_2) &= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot 2 = 2 \\
\text{Area}(R_3) &= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot 2 = 2
\end{aligned}$$

Notice that geometric area is always positive, even for the region below the axis!

Now, to find the **Definite Integral** (Net Signed Area), we treat regions above the x-axis as positive and regions below as negative:

$$\begin{aligned}
\int_0^6 f(x)\,dx &= (\text{Area } R_1) + (\text{Area } R_2) - (\text{Area } R_3) \\
&= 4 + 2 - 2 \\
&= 4
\end{aligned}$$

Finally, to find the **Total Area** (Total Distance), we sum the absolute values of all areas:

$$\begin{aligned}
\text{Total Area} &= \int_0^6 |f(x)|\,dx \\
&= |\text{Area } R_1| + |\text{Area } R_2| + |\text{Area } R_3| \\
&= 4 + 2 + 2 \\
&= 8
\end{aligned}$$

You can check this result by noticing that the positive triangle R_2 and the negative triangle R_3 have equal areas and cancel each other out in the definite integral, leaving only the area of the rectangle R_1.

Practice Exercises

Evaluate the integrals and answer the conceptual questions

24. Converting Limits to Integrals: Which of the following definite integrals is equivalent to the limit given below?

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{3i}{n}\right)^2 \left(\frac{3}{n}\right)$$

1. $\displaystyle\int_4^7 x^2\, dx$
2. $\displaystyle\int_0^3 (4+x)^2\, dx$
3. $\displaystyle\int_0^3 x^2\, dx$
4. Both (a) and (b)

25. Geometric Evaluation: Sketch the region corresponding to the definite integral and evaluate it using a geometric formula.

$$\int_{-3}^{3} \sqrt{9 - x^2}\, dx$$

26. Net Signed Area: Evaluate the definite integral $\int_0^6 (x-2)\,dx$ by interpreting it in terms of areas. Explain why the answer is positive or negative.

27. Total Area vs. Net Area: Consider the function $f(x) = 2x - 6$ on the interval $[0, 5]$.

1. Evaluate the definite integral (Net Signed Area): $\int_0^5 (2x-6)\,dx$
2. Evaluate the total geometric area (Total Distance): $\int_0^5 |2x-6|\,dx$

28. Graphing and Integration: The graph of the function f consists of a semicircle of radius 2 centered at $(2, 0)$ and a line segment from $(4, 0)$ to $(6, 2)$. Evaluate $\int_0^6 f(x)\,dx$.

29. Particle Motion Application: A particle moves along the x-axis with velocity $v(t) = 4 - 2t$ for $0 \leq t \leq 4$.

1. Find the displacement of the particle from $t = 0$ to $t = 4$.
2. Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

4.3.3 Properties of Definite Integrals

The definition of the definite integral initially assumes $a < b$. However, it is convenient to extend this to cover cases where $a = b$ or $a > b$.

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then we define:

$$\int_a^a f(x)\,dx =$$

2. If f is integrable on $[a, b]$, then we define the reversal of limits as:

$$\int_b^a f(x)\,dx =$$

Example 4.16: Evaluate the following:

1. $\displaystyle\int_\pi^\pi \sin(x)\,dx$

2. Given $\displaystyle\int_0^3 (x+2)\,dx = 10.5$, evaluate $\displaystyle\int_3^0 (x+2)\,dx$.

Theorem 4.6: Additive Interval Property

If f is integrable on the three closed intervals determined by $a, b,$ and c, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This theorem tells us that the area of a large region is simply the sum of the areas of its smaller parts.

Theorem 4.7: Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is constant, then the functions of kf and $f \pm g$ are integrable on $[a, , b]$, and

1. $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (Constant Multiple Rule)
2. $\displaystyle\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ (Sum and Difference Rule)

Example 4.17: If $\int_0^5 f(x)dx = 10$ and $\int_0^5 g(x)dx = 3$, find $\int_0^5 [3f(x) - g(x)]dx$.

Theorem 4.8: Preservation of Inequality

1. If f is integrable and nonnegative ($f(x) \geq 0$) on $[a, b]$, then:

$$0 \leq \int_a^b f(x)\, dx$$

2. If f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then:

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

Example 4.18: Without evaluating the integral, explain why $\int_0^1 \sqrt{1+x^2}\,dx \geq \int_0^1 x\,dx$.

Practice Exercises

Use the properties of definite integrals to solve

30. Given $\displaystyle\int_0^5 f(x)\,dx = 10$ and $\displaystyle\int_0^5 g(x)\,dx = -3$, evaluate $\displaystyle\int_0^5 [2f(x) - 4g(x)]\,dx$.

31. Given $\displaystyle\int_{-2}^2 h(x)\,dx = 4$ and $\displaystyle\int_2^5 h(x)\,dx = 3$, find the value of $\displaystyle\int_{-2}^5 h(x)\,dx$.

32. Given $\displaystyle\int_1^7 f(x)\,dx = 15$ and $\displaystyle\int_1^4 f(x)\,dx = 6$, find the value of $\displaystyle\int_4^7 f(x)\,dx$.

33. Write the following expression as a single definite integral of the form $\int_a^b f(x)\,dx$:

$$\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx + \int_6^2 f(x)\,dx$$

34. If $\int_2^6 f(x)\,dx = 8$, evaluate $\int_2^6 (f(x) + 5)\,dx$.
(Hint: Recall that $\int_a^b k\,dx$ represents the area of a rectangle with height k and width $b - a$).

35. Determine the value of the integral $\int_5^5 (\sin(x^2) + e^x)\,dx$.

4.4 The Fundamental Theorem of Calculus

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Evaluate a definite integral using the Fundamental Theorem of Calculus.	I can evaluate a definite integral by finding an antiderivative and applying the Fundamental Theorem of Calculus.
Understand and use the Mean Value Theorem for Integrals.	I can apply the Mean Value Theorem for Integrals to find an average value guaranteed on an interval.
Find the average value of a function over a closed interval.	I can compute the average value of a function on a closed interval using a definite integral.
Understand and use the Second Fundamental Theorem of Calculus.	I can differentiate an accumulation function using the Second Fundamental Theorem of Calculus.

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus**.

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in the figure below. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.

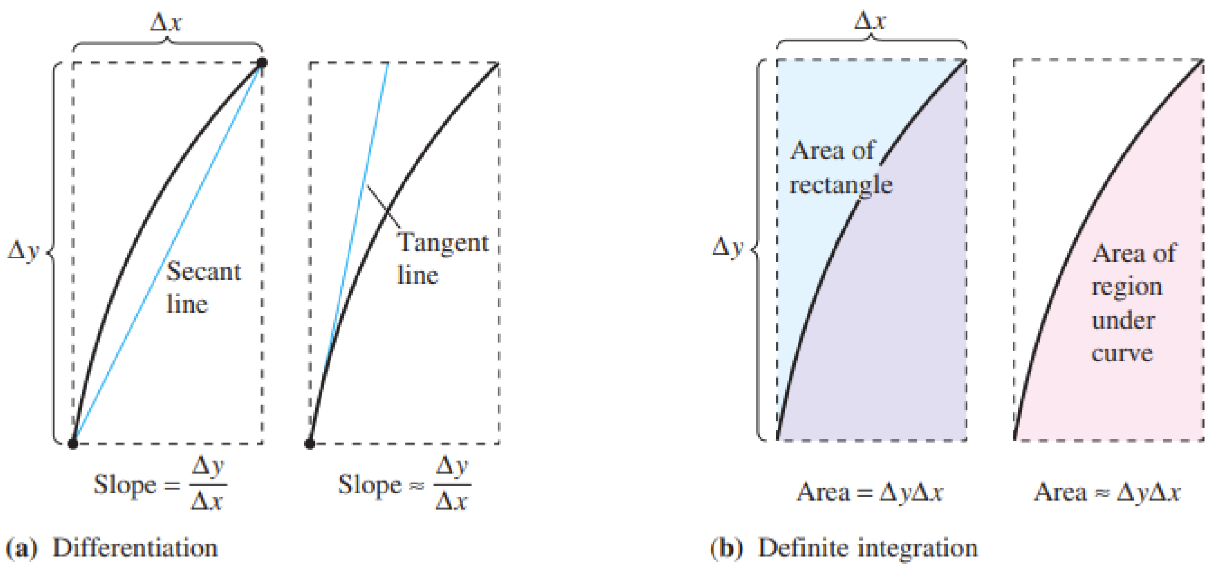


Figure 4.7 Differentiation and definite integration have an "inverse" relationship.

Theorem 4.9: The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof: The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be the following partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This important equation tells you that by applying the Mean Value Theorem you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$. Taking the limit (as $||\Delta|| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)dx$$

□

Guidelines for Using the Fundamental Theorem of Calculus

1. Provided that you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.

2. When applying the Fundamental Theorem of Calculus the following notation, an evaluation bar, is convenient.

$$\begin{aligned} \int_a^b f(x)dx &= F(x)\Big|_a^b \\ &= F(b) - F(a) \end{aligned}$$

   For example,

$$\begin{aligned} \int_1^3 x^3 dx &= \frac{x^4}{4}\Big|_1^3 \\ &= \frac{3^4}{4} - \frac{1^4}{4} \\ &= \frac{81}{4} - \frac{1}{4} \\ &= 20 \end{aligned}$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned} \int_a^b f(x)dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

   Notice that you can also use brackets as an evaluation bar.

Example 4.19: Evaluate each definite integral.

1. $\displaystyle\int_1^2 (x^2 - 3)\,dx$

2. $\displaystyle\int_1^4 3\sqrt{x}\,dx$

3. $\displaystyle\int_0^{\pi/4} \sec^2(x)\,dx$

Example 4.20: Evaluate $\displaystyle\int_0^2 |2x-1|\,dx$.

Hint: Rewrite the integrand using the definition of absolute value to split the integral into two parts.

$$|2x-1| = \begin{cases} \qquad\qquad , & x < \frac{1}{2} \\ \qquad\qquad , & x \geq \frac{1}{2} \end{cases}$$

Practice Exercises

Evaluate the following definite integrals

36. $\displaystyle\int_{-1}^{2} (3x^2 - 2x + 1)\, dx$

37. $\displaystyle\int_{1}^{8} \sqrt[3]{x}\, dx$

38. $\displaystyle\int_{0}^{\pi} (2 + \cos(x))\, dx$

39. $\displaystyle\int_{0}^{4} |x - 3|\, dx$

40. $\displaystyle\int_{1}^{2} \frac{3x^2 + 5}{x^2}\, dx$

41. Let $F(x)$ be an antiderivative of $f(x)$. If $\int_2^5 f(x)\,dx = 12$ and $F(5) = 18$, find the value of $F(2)$.

4.4.1 Mean Value theorem for Integrals

In Section 4.2.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve.

Theorem 4.10: Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Proof: **Case 1:** If f is constant on the interval $[a, b]$, the theorem is clearly valid because c can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Because $f(m) \leq f(x) \leq f(M)$ for all x in $[a, b]$, you can apply Theorem 4.8 to write the following.

$$\begin{aligned}
\int_a^b f(m)dx &\leq \int_a^b f(x)dx \leq \int_a^b f(M)dx \\
f(m)(b-a) &\leq \int_a^b f(x)dx \leq f(M)(b-a) \\
f(m) &\leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(M)
\end{aligned}$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \quad \text{or} \quad f(c)(b-a) = \int_a^b f(x)dx$$

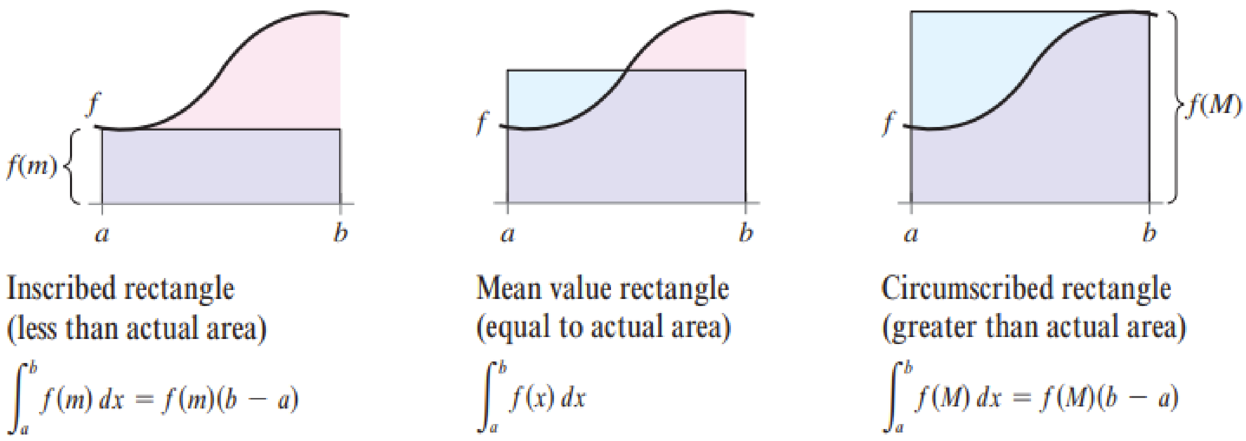


Figure 4.8

Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval.

□

4.4.2 Average Value of a Function

The value $f(c)$ in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is:

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)dx$$

Derivation (Why this works): If we partition $[a, b]$ into n subintervals, the arithmetic average of the function values is:

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \cdots + f(c_n)\right]$$

By manipulating this with $\Delta x = \frac{b-a}{n}$, we get:

$$a_n = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x$$

Taking the limit as $n \to \infty$ turns the sum into an integral:

$$\lim_{n\to\infty} a_n = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 4.21: Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Practice Exercises

Find the average value and apply the Mean Value Theorem

42. Find the average value of the function $f(x) = x^3 - x$ on the interval $[0, 2]$.

43. Find the average value of $g(x) = \sin(x)$ on the interval $[0, \pi]$.

44. Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = 4 - x^2$ on the interval $[0, 2]$.

45. The temperature (in °F) in a room from 9 AM to 2 PM is modeled by the function $T(t) = 70 + 2\sqrt{t}$, where t is measured in hours starting at 9 AM ($t = 0$). Find the average temperature of the room during this 5-hour period.

46. Given that the average value of a continuous function f on the interval $[2, 6]$ is 12, evaluate the definite integral $\int_2^6 f(x)\,dx$.

47. Find the average value of $h(x) = e^{2x}$ on the interval $[0, \ln(3)]$.

4.4.3 The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used as the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration.

The Definite Integral as a Number

$$\int_a^b f(x)\,dx$$

- a is a constant
- b is a constant
- f is a function of x

The Definite Integral as a Function of x

$$F(x) = \int_a^x f(t)\,dt$$

- a is a constant
- F is a function of x
- f is a function of t

Example 4.22: Evaluate the function $F(x) = \int_0^x \cos(t)dt$ at $x = 0, \pi/6, \pi/4, \pi/2$.

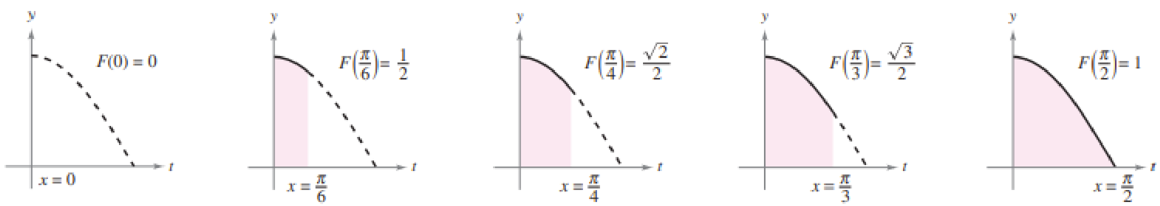


Figure 4.9 $F(x) = \int_0^x \cos(t)dt$ is the area under the curve $f(t) = \cos(t)$ from 0 to x.

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos(t)$ from $t = 0$ to $t = x$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

Notice that in the previous example, the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin(x)] = \frac{d}{dx}\left[\int_0^x \cos(t)dt\right] = \cos(x)$$

This result is generalized in the following theorem, the **Second Fundamental Theorem of Calculus**.

Theorem 4.11: The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Proof: Define $F(x) = \int_a^x f(t)dt$. By the definition of the derivative:

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)dt\right] \end{aligned}$$

By the Mean Value Theorem for Integrals, there exists a c in $[x, x+\Delta x]$ such that the integral equals $f(c)\Delta x$.

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} f(c)\Delta x\right] \\ &= \lim_{\Delta x \to 0} f(c) \\ &= f(x) \quad (\text{since } c \to x \text{ as } \Delta x \to 0) \end{aligned}$$

□

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function.

Example 4.23: Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}dt\right]$.

Applying the Chain Rule If the upper limit is not just x but a function $u(x)$, we must apply the Chain Rule.

$$\frac{d}{dx}\left[\int_a^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x)$$

Example 4.24: Find the derivative of $F(x) = \displaystyle\int_{\pi/2}^{x^3} \cos(t)dt$.

Practice Exercises

Apply the Second Fundamental Theorem of Calculus

48. Evaluate the derivative: $\dfrac{d}{dx}\left[\displaystyle\int_3^x \sqrt{t^2+4}\,dt\right]$.

49. Find $F'(x)$ given $F(x) = \displaystyle\int_1^{x^3} \frac{1}{t^2+1}\,dt$.

50. Find $\frac{dy}{dx}$ for the function $y = \int_x^5 \cos(t^2)\,dt$.
(Hint: Switch the limits of integration first.)

51. Let $g(x) = \int_0^{2x} f(t)\,dt$. If $f(t) = e^t + t$, find $g'(0)$.

52. Find the equation of the tangent line to the graph of $F(x) = \int_1^x \sqrt{t^3 + 8}\,dt$ at the point where $x = 2$.

53. Let $H(x) = \int_0^x f(t)\,dt$, where f is the continuous function defined by $f(t) = t^2 - 4$.

1. Find $H'(x)$.
2. Find $H''(x)$.
3. On what interval(s) is $H(x)$ concave up?

4.5 Integration By substitution

Lesson Objectives & Success Criteria

Key Topics & Formulas	Success Criteria
Use pattern recognition to find an indefinite integral.	I can recognize patterns that match known derivatives and use them to find indefinite integrals.
Use a change of variables to find an indefinite integral.	I can use a change of variables to rewrite and evaluate an indefinite integral.
Use the General Power Rule for Integration to find an indefinite integral.	I can apply the General Power Rule to find indefinite integrals involving powers of x.
Use a change of variables to evaluate a definite integral.	I can use a change of variables to evaluate a definite integral and correctly adjust the limits of integration.
Evaluate a definite integral involving an even or odd function.	I can use symmetry and properties of even and odd functions to simplify and evaluate definite integrals.

4.5.1 Pattern Recognition

The role of substitution in integration is comparable to the role of the **Chain Rule** in differentiation.

Recall:

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

Thus, for integration:

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

Theorem 4.12: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

If we let $u = g(x)$, then $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Visualizing the Pattern: The composite function has an *outside function* f and an *inside function* g. The derivative $g'(x)$ must be present as a factor.

Outside Function

Derivative of Inside

$$\int \overset{\downarrow}{f}\big(\underset{\uparrow}{g}(x)\big)\overbrace{g'(x)}\,dx = F(g(x)) + C$$

Inside Function

Example 4.25: Find $\int (x^2 + 1)(2x)dx$.

Example 4.26: Find $\int 5\cos(5x)dx$.

Often, the integrand contains the variable part of $g'(x)$ but is missing a constant multiple. We can fix this using the Constant Multiple Rule.

Example 4.27: Find $\int x(x^2+1)^2dx$.

4.5.2 Change of Variables

With a formal **change of variables**, you completely rewrite the integral in terms of u and du. This is useful for complicated integrands where pattern recognition is difficult.

If $u = g(x)$, then $du = g'(x)dx$, and the integral becomes:

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

Example 4.28: Find $\int \sqrt{2x-1}\,dx$.

Sometimes, simply replacing $g(x)$ with u leaves extra x's in the integrand. In these cases, you must solve the substitution equation for x to rewrite the entire integral in terms of u.

Example 4.29: Find $\int x\sqrt{2x-1}\,dx$.

Step 1: Let $u = 2x - 1$ and find du.

Step 2: Solve $u = 2x - 1$ for x.

Step 3: Substitute and Integrate.

Example 4.30: Find $\int \sin^2(3x)\cos(3x)\,dx$.

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the **inner part** of a composite function.
2. Compute $du = g'(x)dx$.
3. Rewrite the integral **entirely** in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Practice Exercises

Find the indefinite integral

54. $\int (5x+1)^4\,dx$

55. $\int x(x^2-9)^3\,dx$

56. $\int \frac{x^2}{\sqrt{x^3+8}}\,dx$

57. $\int \cos^3(2x)\sin(2x)\,dx$

58. $\int x\sqrt{x-4}\,dx$

59. $\int \frac{x}{\sqrt{2x+1}}\,dx$

4.5.3 The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration**.

Theorem 4.13: The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

Example 4.31: Identify u and du, then evaluate the integral.

1. $\displaystyle\int 3(3x-1)^4\,dx$

2. $\displaystyle\int (2x+1)(x^2+x)\,dx$

3. $\displaystyle\int 3x^2\sqrt{x^3-2}\,dx$

4. $\displaystyle\int \frac{-4x}{(1-2x^2)^2}\,dx$

5. $\displaystyle\int \cos^2 x \sin x\,dx$

4.5.4 Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than converting back to x.

Theorem 4.14: Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Key Step: You must change the limits!

- Lower limit: $x = a \Rightarrow u = g(a)$
- Upper limit: $x = b \Rightarrow u = g(b)$

Example 4.32: Evaluate $\int_0^1 x(x^2+1)^3 dx$.

Step 1: Choose u and find du.

Step 2: Change the Limits.

$$x = 0 \Rightarrow u =$$
$$x = 1 \Rightarrow u =$$

Step 3: Evaluate in terms of u.

Note: If you change the limits, you do NOT need to go back to x.

Example 4.33: Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}} dx$.

Hint: Let $u = \sqrt{2x-1}$. This means $u^2 = 2x - 1$. Solve for x and dx.

Geometric Interpretation: The substitution transforms the original region into a new region with a different shape, but the **area remains the same**.

$$\text{Area under } f(x) \text{ on } [1,5] = \text{Area under } g(u) \text{ on } [1,3]$$

Practice Exercises

Evaluate the definite integrals using the method of change of variables

60. $\displaystyle\int_0^2 3x^2(x^3+1)^3\,dx$

61. $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx$

62. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

63. $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$

64. Rewrite the integral $\displaystyle\int_0^2 e^{2x}(1+e^{2x})^5\,dx$ in terms of u, where $u = 1+e^{2x}$. Determine the new limits of integration, but do not evaluate the final result.

65. $\displaystyle\int_1^e \frac{\ln(x)}{x}\,dx$

4.5.5 Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over a symmetric interval $[-a, a]$ by recognizing the function's symmetry.

Definition of Even and Odd Functions

The function $y = f(x)$ is **even** if $f(-x) = f(x)$. (Symmetric about the y-axis)
The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$. (Symmetric about the origin)

Theorem 4.15: Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an **even** function, then:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

2. If f is an **odd** function, then:

$$\int_{-a}^{a} f(x)dx = 0$$

Example 4.34: Evaluate $\int_{-\pi/2}^{\pi/2} \left(\sin^3(x)\cos(x) + \sin(x)\cos(x)\right) dx$.

Step 1: Check for Symmetry. Let $f(x) = \sin^3(x)\cos(x) + \sin(x)\cos(x)$. Test $f(-x)$.

Step 2: Apply the Theorem.

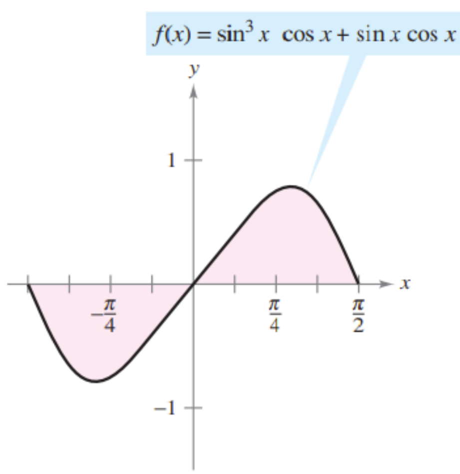


Figure 4.10 Geometric cancellation of signed area for an odd function.

4.6 Numerical Integration

Lesson Objectives & Success Criteria

Key Topics & Formulas	Success Criteria
Approximate a definite integral using the Midpoint Rule.	I can approximate a definite integral using the Midpoint Rule and interpret the result as an estimate of area.
Approximate a definite integral using the Trapezoidal Rule.	I can approximate a definite integral using the Trapezoidal Rule and explain how it estimates area using trapezoids.
Approximate a definite integral using Simpson's Rule.	I can approximate a definite integral using Simpson's Rule and apply the correct formula for an even number of subintervals.
Analyze the approximate errors in the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule.	I can analyze and compare the errors of the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule using concavity and known error behavior.

I deviate from the textbook slightly to include the Midpoint Rule and a small discussion. The theorem numbers will still match the textbook.

Now that we have discussed the Fundamental Theorem of Calculus, you might have asked yourself 'why would I use a Riemann sum if I can just take the antiderivative?'

First, some elementary functions do not have antiderivatives that are elementary functions. For example, there is no elementary function that has any of the following functions as its derivative

$$\sqrt[3]{x}\sqrt{1-x}, \quad \sqrt{x}\cos(x), \quad \frac{\cos(x)}{x}, \quad \sqrt{1-x^3}, \quad \sin(x^2), \quad e^{-x^2}$$

If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, the Fundamental Theorem of Calculus cannot be applied, and you must resort to an approximation technique.

Second, almost every integral is done numerically, even when it can be done symbolically (analytically) in a closed form way. Symbolic calculators, like Wolfram Alpha, will perform the necessary antiderivative. However, in practice, most of the time we just want a number. Performing the actual antiderivative is a complex process, whereas evaluating a function and multiplying is something a *computer* is designed to do. *Numerical Quadrature* (or numerical integration) is a way of calculating the integral approximately, but in a way that is "close enough".

4.6.1 The Midpoint Rule

The Midpoint rule is almost identical to the Left and Right Riemann Sum, except now it will shift the interval to be in between the left and right points. Recall the definition of a Riemann Sum.

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

For each subinterval $[x_{i-1}, x_i]$, calculate the midpoint m_i using the formula

$$m_i = \frac{x_{i-1} + x_i}{2}$$

Then form a rectangle for each subinterval. The area of the ith rectangle is

$$A = f(m_i)\Delta x$$

This implies that the sum of the areas of the n rectangles is

$$\begin{aligned} A &= f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x \\ &= \Delta x[f(m_1) + f(m_2) + \cdots + f(m_n)] \end{aligned}$$

Since f is continuous, the limit of this Riemann Sum as $n \to \infty$ is precisely the definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \int_a^b f(x)dx$$

Theorem: The Midpoint Rule

Let f be continuous on $[a, b]$. The Midpoint Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.35: Use the Midpoint Rule to estimate $\displaystyle\int_0^1 x^2 dx$ using four subintervals.

Start by computing the width of each subinterval. $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$. Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}\frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right]$$

The midpoints of each of these subintervals are halfway in between each value. For example, $\dfrac{\frac{1}{4} - 0}{2} = \dfrac{1}{8}$. Therefore, the midpoints of the subintervals are $m_i = \left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned}
M_4 &= \sum_{i=1}^{n} f(m_i)\Delta x \\
&= \sum_{i=1}^{n} f(m_i)\left(\frac{1}{4}\right) \\
&= f\left(\frac{1}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{3}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{5}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{7}{8}\right)\cdot\left(\frac{1}{4}\right) \\
&= \left(\frac{1}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{9}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{25}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{49}{64}\right)\cdot\left(\frac{1}{4}\right) \\
&= \left(\frac{1}{4}\right)\left[\left(\frac{1}{64}\right) + \left(\frac{9}{64}\right) + \left(\frac{25}{64}\right) + \left(\frac{49}{64}\right)\right] \\
&= \left(\frac{1}{4}\right)\left(\frac{21}{16}\right) \\
&= \frac{21}{64}
\end{aligned}$$

As you can see from the graph on the right, the function goes through the middle of each rectangle. So for a function that is increasing and concave up, the rectangles will have a slight overestimate on the left and a slight underestimate on the right.
You can see it for yourself on desmos.
Of course, by increasing the number of rectangles, you get more and more accurate results.

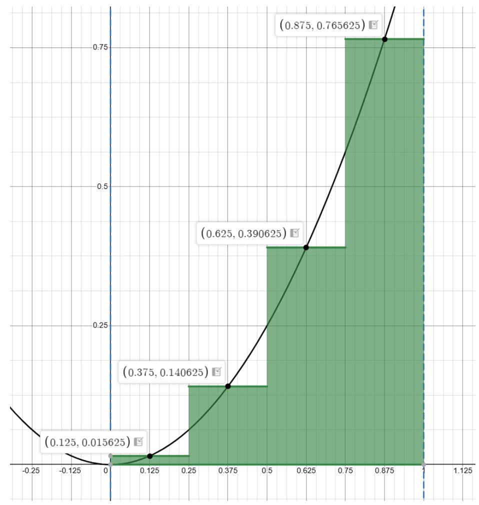


Figure 4.11

4.6.2 The Trapezoidal Rule

Another of to approximate a definite integral is to use n trapezoids, as shown in the figure below.

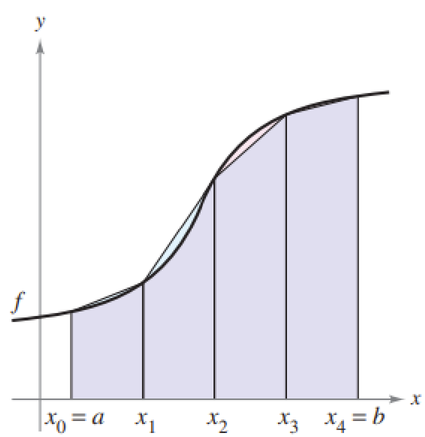


Figure 4.12 The area of a region can be approximated using four trapezoids.

In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Then form a trapezoid for each subinterval. The area of the ith trapezoid is

$$A = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right)$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned} A &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0)+f(x_1)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \end{aligned}$$

Letting $\Delta x = (b-a)/n$, you can take the limit as $n \to \infty$ to obtain

$$\lim_{n\to\infty}\left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$\begin{aligned} &= \lim_{n\to\infty}\left[\frac{[f(a)-f(b)]\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\Delta x\right] \\ &= \lim_{n\to\infty}\frac{[f(a)-f(b)]\Delta x}{2} + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= 0 + \int_a^b f(x)dx \end{aligned}$$

Theorem 4.16: The Trapezoidal Rule

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.36: Use the Trapezoidal Rule to estimate $\displaystyle\int_0^1 x^2\,dx$ using four subintervals.

Start by computing the width of each subinterval.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right].$$

The Trapezoidal Rule uses the endpoints of each subinterval. Thus, the partition points are

$$x_i = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

The trapezoidal rule with n subintervals is given by

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Applying this formula with $n = 4$, we have

$$\begin{aligned}
T_4 &= \frac{1}{2}\left(\frac{1}{4}\right)\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \\
&= \frac{1}{8}\left[0 + 2\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{9}{16}\right) + 1\right] \\
&= \frac{1}{8}\left[\frac{2}{16} + \frac{2}{4} + \frac{18}{16} + 1\right] \\
&= \frac{1}{8}\left(\frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 1\right) \\
&= \frac{1}{8}\left(\frac{22}{8}\right) \\
&= \frac{11}{32}.
\end{aligned}$$

As you can see from the graph on the right, we can form little trapezoids to approximate the function. In this case, since the function is increasing concave up, every estimate will be a slight overestimate.
You can see it for yourself on desmos.
Of course, by increasing the number of rectangles, you get more and more accurate results.

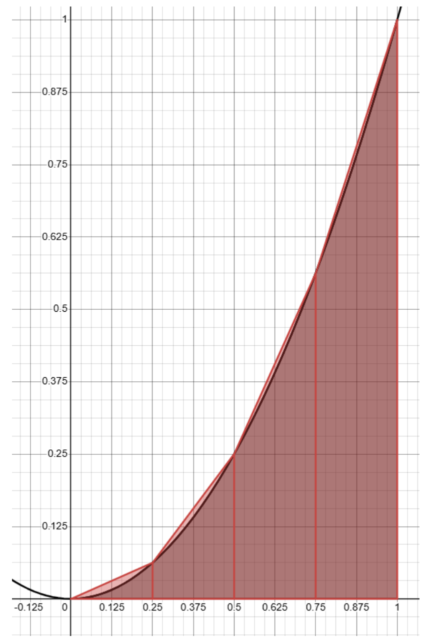


Figure 4.13

4.6.3 Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate $f$by a *first*-degree polynomial (that is, a line). In Simpson's Rule, named after the English mathematician Thomas Simpson (1710–1761), you take this procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, first we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

Theorem 4.17: Integral of $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{b-a}{2}\right) + p(b)\right]$$

Proof:

$$\begin{aligned}
\int_a^b p(x)\,dx &= \int_a^b (Ax^2+Bx+C)\,dx \\
&= \left[\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right]_a^b \\
&= \frac{A(b^3-a^3)}{3}+\frac{B(b^2-a^2)}{2}+C(b-a) \\
&= \left(\frac{b-a}{6}\right)\big[2A(a^2+ab+b^2)+3B(b+a)+6C\big].
\end{aligned}$$

By expansion and collection of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2+Ba+C)}_{p(a)}+\underbrace{4\left[A\left(\frac{a+b}{2}\right)^2+B\left(\frac{a+b}{2}\right)+C\right]}_{4p\left(\frac{a+b}{2}\right)}+\underbrace{(Ab^2+Bb+C)}_{p(b)},$$

Therefore,

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right].$$

□

To develop Simpson's Rule for approximating a definite integral, you again partition the subinterval $[a,b]$ into n subintervals, each of width $\Delta x=(b-a)/n$. This time, whoever, n is required to be even, and the subintervals are grouped in pairs such that

$$a=\underbrace{x_0<x_1<x_2}_{[x_0,x_2]}\underbrace{<x_3<x_4}_{[x_2,x_4]}<\cdots<\underbrace{x_{n-2}<x_{n-1}<x_n}_{[x_{n-2},x_n]}=b.$$

On each (double) subinterval $[x_{i-2},x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2. For example, on the subinterval $[x_0,x_2]$, you can choose the polynomial of least degree passing through the points $(x_0,y_0),(x_1,y_1)$, and (x_2,y_2). Now, using p as an approximation of f on this subinterval, you have, by Theorem 4.17,

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx &= \left(\frac{x_2-x_0}{6}\right)\left[p(x_0)+4p\left(\frac{x_2-x_0}{2}\right)+p(x_2)\right] \\
&= \frac{2[(b-a)/n]}{6}[p(x_0)+4p(x_1)+p(x_2)] \\
&= \frac{b-a}{3n}[f(x_0)+4f(x_1)+f(x_2)]
\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

Theorem 4.18: Simpson's Rule (n is even)

Let f be continuous on $[a, b]$. Simpson's Rule for approximating $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]$$

Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.37: Use Simpson's Rule to estimate $\displaystyle\int_0^1 x^2\,dx$ using four subintervals.

Start by computing the width of each subinterval.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right].$$

Simpson's Rule requires an even number of subintervals, which is satisfied here. The partition points are

$$x_i = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

The Simpson's Rule formula with n subintervals is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Applying this formula with $n = 4$, we obtain

$$\begin{aligned}
S_4 &= \frac{1}{3}\left(\frac{1}{4}\right)\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] \\
&= \frac{1}{12}\left[0 + 4\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{9}{16}\right) + 1\right] \\
&= \frac{1}{12}\left[\frac{4}{16} + \frac{2}{4} + \frac{36}{16} + 1\right] \\
&= \frac{1}{12}\left(\frac{1}{4} + \frac{1}{2} + \frac{9}{4} + 1\right) \\
&= \frac{1}{12}(4) \\
&= \frac{1}{3}.
\end{aligned}$$

Now, however, it should be obvious that this will be exact. This is because we are approximating a degree 2 polynomial with a degree 2 polynomial. However, I wanted to provide the same function in each example so the work could be consistently displayed.
You can see it for yourself on desmos.

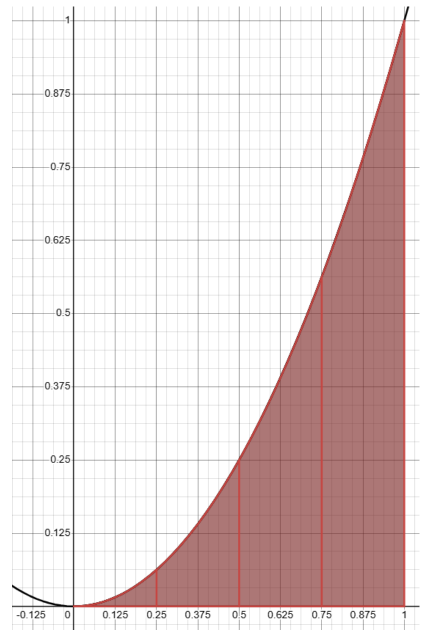


Figure 4.14

Here is a different example where the approximation is not exact.

Example 4.38: Use Simpson's Rule to estimate $\int_0^\pi \sin(x)\,dx$ using four subintervals.
Start by computing the width of each subinterval.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{\pi}{4}\right], \quad \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \quad \left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \quad \left[\frac{3\pi}{4}, \pi\right].$$

The partition points are

$$x_i = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}.$$

Simpson's Rule formula with n subintervals is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right].$$

Applying this formula with $n = 4$, we have

$$\begin{aligned}
S_4 &= \frac{\pi/4}{3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \\
&= \frac{\pi}{12}\left[0 + 4\cdot\frac{\sqrt{2}}{2} + 2\cdot 1 + 4\cdot\frac{\sqrt{2}}{2} + 0\right] \\
&= \frac{\pi}{12}\left[4\cdot\frac{\sqrt{2}}{2} + 2 + 4\cdot\frac{\sqrt{2}}{2}\right] \\
&= \frac{\pi}{12}\left[4\cdot\frac{\sqrt{2}}{2} + 4\cdot\frac{\sqrt{2}}{2} + 2\right] \\
&= \frac{\pi}{12}\left[4\sqrt{2} + 2\right] \\
&= \frac{\pi}{12}(2 + 4\sqrt{2}) \\
&= \frac{\pi}{6}(1 + 2\sqrt{2}) \\
&\approx 2.005
\end{aligned}$$

The approximation show a slight underestimation near the top, but an overestimation on the sides.
You can see it for yourself on desmos.

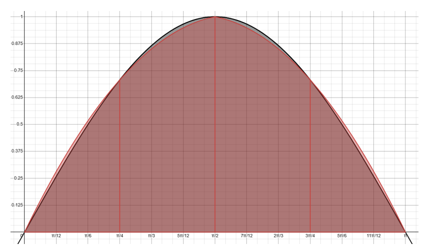


Figure 4.15

4.6.4 Error Analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. The following theorem, which is listed without proof, gives the formulas for estimating the errors involved in use of the Midpoint Rule, Simpson's Rule, and the Trapezoidal Rule.

Theorem 4.19: Errors in the Midpoint Rule, Simpson's Rule, and the Trapezoidal Rule.

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by the Midpoint Rule is

$$E \leq \frac{(b-a)^3}{24n^2}[\max|f''(x)|], \quad a \leq x \leq b$$

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by the Trapezoidal Rule is

$$E \leq \frac{(b-a)^3}{12n^2}[\max|f''(x)|], \quad a \leq x \leq b$$

If f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by Simpson's Rule is

$$E \leq \frac{(b-a)^5}{180n^4}[\max|f^{(4)}(x)|], \quad a \leq x \leq b$$

Theorem 4.19 states that the errors generated by the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$ and $f^{(4)}(x)$ in the interval $[a, b]$ Furthermore, these errors can be made arbitrarily small by increasing n, provided that the functions are continuous and therefore bounded in $[a, b]$.

Practice Exercises

Approximating Definite Integrals

66. Use the Midpoint Rule with $n = 4$ subintervals to approximate the integral $\displaystyle\int_0^4 \sqrt{x^2+1}\,dx$. Round your answer to three decimal places.

67. Use the Trapezoidal Rule with $n = 4$ to approximate the value of $\displaystyle\int_1^3 \frac{1}{x}\,dx$. Show the expansion of the sum.

68. The function f is continuous, positive, and concave up on the interval $[a, b]$.

1. Does the Trapezoidal Rule approximation for $\int_a^b f(x)\,dx$ yield an overestimate or an underestimate? Explain geometrically.

2. Does the Midpoint Rule approximation for $\int_a^b f(x)\,dx$ yield an overestimate or an underestimate?

69. Selected values of a continuous function f are given in the table below.

x	0	2	4	6	8
$f(x)$	12	7	5	8	10

1. Use the Trapezoidal Rule with 4 subintervals of equal length to approximate $\int_0^8 f(x)\,dx$.

2. Use the Midpoint Rule with 2 subintervals of equal length to approximate $\int_0^8 f(x)\,dx$.

70. A radar gun measures the velocity $v(t)$ (in ft/sec) of a car at 5-second intervals.

t (sec)	0	5	10	15	20
$v(t)$	0	30	50	70	80

Estimate the total distance traveled by the car from $t = 0$ to $t = 20$ seconds using the Trapezoidal Rule with 4 subintervals.

71. Consider the integral $I = \displaystyle\int_0^2 e^{-x^2}\,dx$. If T_n represents the Trapezoidal Rule approximation and M_n represents the Midpoint Rule approximation with n subintervals, arrange T_4, M_4, and I in increasing order. (Note: Consider the concavity of $f(x) = e^{-x^2}$).

72. (Simpson's Rule) Use Simpson's Rule with $n = 4$ subintervals to approximate $\int_0^4 x^3\,dx$. Compare your result with the exact value of the integral.

73. (Simpson's Rule) The table below shows values of the function $g(x)$.

x	1	2	3	4	5	6	7
$g(x)$	2	5	8	6	4	7	3

Approximate $\int_1^7 g(x)\,dx$ using Simpson's Rule with $n = 6$ subintervals.