Chapter 4 Integration

Teacher Notes

By: Cole ℝidgway

Contents

4.1 Antiderivatives and Indefinite Integration

Lesson Objectives & Success Criteria

Key Topics & Formulas	Success Criteria
Write the general solution of a differential equation.	I can write the general solution of a differential equation, including the constant of integration.
Use indefinite integral notation for antiderivatives.	I can correctly use indefinite integral notation to represent families of antiderivatives.
Use basic integration rules to find antiderivatives.	I can apply basic integration rules to find antiderivatives of elementary functions.
Find a particular solution of a differential equation.	I can find a particular solution of a differential equation by using given initial conditions.

4.1.1 Antiderivatives

We have already discussed antiderivatives in this class. However, in this chapter we will formula define them and explore the concept in more detail.

Definition of an Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative, rather than *the* antiderivative of f. This is because for any constant C, the function given by the F is an antiderivative of f.

Theorem 4.1: Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form:

$$G(x) = F(x) + C$$

for all x in I where C is a constant.

Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $\frac{d}{dx}[x^2] = 2x$ you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C$$

where C is a constant. The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f. and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.

Example 4.1: Find the general solution to the differential equation $y' = 2$.
To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C$$

4.1.2 Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)dx$$

The operation of finding all solution of this equation is called **antidifferentiation** or **indefinite integration** and is denoted by the integral sign $\int$. The general solution is denoted by

$$y = \int f(x)dx = F(x) + C$$

The expression $\int f(x)dx$ is read as the *indefinite integral of f with respect to x.* So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

4.1.3 Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)dx = F(x) + C$$

Moreover, if $\int f(x)dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following:

Basic Integration Rules

Differentiation Formula	**Integration Formula**
$\frac{d}{dx}[C] = 0$	$\int 0\, dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\, dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\, dx = k\int f(x)\, dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\, dx = -\csc x + C$

Example 4.2: Describe the antiderivative of $3x$.
Begin by writing

$$\begin{aligned}
\int 3x\,dx &= 3\int x\,dx && \text{Constant Multiple Rule.}\\
&= 3\int x^1\,dx && \text{Rewrite } x \text{ as } x^1.\\
&= 3\left(\frac{x^2}{2}\right) + C && \text{Power Rule.}\\
&= \frac{3}{2}x^2 + C && \text{Simplify.}
\end{aligned}$$

Recall that we can rewrite functions before integrating to get them into more familiar forms.

Example 4.3:

1.

$$\begin{aligned}
\int \frac{1}{x^3}dx &= \int x^{-3}dx\\
&= \frac{x^{-2}}{-2} + C\\
&= -\frac{1}{2x^2} + C
\end{aligned}$$

2.

$$\begin{aligned}
\int \sqrt{x}\,dx &= \int x^{1/2}dx\\
&= \frac{x^{3/2}}{3/2} + C\\
&= \frac{2}{3}x^{3/2} + C
\end{aligned}$$

Remember that you can check your answer to an antidifferentiation problem by differentiating.

The basic integration rules listed earlier in this section allow you to integrate any polynomial function, as shown in the next example.

Example 4.4:

$$\begin{aligned}\int (3x^4 - 5x^2 + x)dx &= \int 3x^4 dx + \int -5x^2 dx + \int x dx \\ &= 3\int x^4 dx - 5\int x^2 dx + \int x dx \\ &= 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C \\ &= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C\end{aligned}$$

There are several ways of rewriting integrals that may be useful. Separating a fraction or rewriting as a product, as the next two examples show.

Example 4.5:

$$\begin{aligned}\int \frac{x+1}{\sqrt{x}}dx &= \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx \\ &= \int (x^{1/2} + x^{-1/2})dx \\ &= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C \\ &= \frac{2}{3}x^{3/2} + 2x^{1/2} + C \\ &= \frac{2}{3}\sqrt{x}(x+3) + C\end{aligned}$$

Example 4.6:

$$\begin{aligned}\int \frac{\sin(x)}{\cos^2(x)}dx &= \int \left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) dx \\ &= \int \sec(x)\tan(x)dx \\ &= \sec(x) + C\end{aligned}$$

Practice Exercises

Evaluate the following indefinite integrals

1. $\displaystyle\int (3x^4 - 5x^2 + 2)\,dx$

2. $\displaystyle\int \left(\sqrt[3]{x} + \frac{1}{x^3}\right)\,dx$

3. $\displaystyle\int \frac{x^2 + 2x - 3}{\sqrt{x}}\,dx$

4. $\displaystyle\int (x+2)(2x-3)\,dx$

5. $\displaystyle\int (4\sin(x) - 3\csc^2(x))\,dx$

6. $\displaystyle\int \frac{\cos(t)}{\sin^2(t)}\,dt$

4.1.4 Initial Conditions and Particular Solutions

We have actually already covered this before, when we did **initial value problems**. It is now, however, that we can write them in terms of the indefinite integral.

Example 4.7: Find the particular solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

that satisfies the initial condition $F(1) = 0$.
Start by integrating:

$$\begin{aligned} F(x) &= \int \frac{1}{x^2} dx \\ &= \int x^{-2} dx \\ &= \frac{x^{-1}}{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$

Then calculate C using the initial condition.

$$F(1) = \frac{-1}{1} + C = 0 \quad \Rightarrow \quad C = 1$$

Thus

$$f(x) = -\frac{1}{x} + 1, \quad x > 0$$

So far in this section you have been using as the x variable of integration. In applications, it is often convenient to use a different variable. For instance, in the following example involving time, the variable of integration is t.

Example 4.8: A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

(a) Find the position function giving the height as s as a function of time t.
Let $t = 0$ represent the initial time. The two given initial conditions can be written as

$s(0) = 80$	Initial height is 80 feet.
$s'(0) = 64$	Initial velocity is 80 feet per second.

Using -32 feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$
$$s'(t) = \int s''(t)dx = \int -32dt = -32t + C_1$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$ you obtain

$$s(t) = \int s'(t)dt = \int (-32t + 64)dt = -16t^2 + 64t + C_2$$

Using the initial height, you get

$$s(0) = 80 = -16(0)^2 + 64(0) + C_2$$

which implies that $C_2 = 80$. So, the position function is

$$s(t) = -16t^2 + 64t + 80$$

(b) When does the ball hit the ground?
Using the position function we just found, you can find the time the ball hits the ground by setting the position function equal to 0.

$$\begin{aligned} s(t) = -16t^2 + 64t + 80 &= 0 \\ -16(t+1)(t-5) &= 0 \\ t &= -1, 5 \end{aligned}$$

Where you can conclude $t = 5$ since time cannot be negative. The ball hits the ground 5 seconds after it was thrown.

Practice Exercises

Solve the following initial value and motion problems

7. Find the particular solution $y = f(x)$ that satisfies the differential equation $f'(x) = 6x^2 - 4x + 2$ and the initial condition $f(1) = 9$.

8. Find the function $y(x)$ if $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and $y(9) = 4$.

9. Find the particular solution for $f(x)$ given the second derivative $f''(x) = 6x$, with initial conditions $f'(0) = 2$ and $f(0) = 5$.

10. A particle moves along the x-axis with a velocity given by $v(t) = 3t^2 - 2t$ for $t \geq 0$. If the particle is at position $x = 3$ when $t = 1$, find the position of the particle at $t = 3$.

11. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 64 feet per second. Use the acceleration due to gravity $a(t) = -32$ ft/sec^2.

1. Find the position function $s(t)$.
2. Determine the maximum height reached by the ball.

12. An object is dropped from a cliff that is 400 feet high. Its acceleration is $a(t) = -32$ ft/sec^2 and initial velocity is $v(0) = 0$. Determine the velocity of the object at the moment it impacts the ground.

4.2 Area

Lesson Objectives & Success Criteria

Key Topics & Formulas	Success Criteria
Use sigma notation to write and evaluate a sum.	I can write a sum using sigma notation and evaluate it correctly.
Understand the concept of area.	I can explain area as an accumulation of infinitely many small regions.
Approximate the area of a plane region.	I can approximate the area of a plane region using rectangles and sums.
Find the area of a plane region using limits.	I can find the exact area of a plane region by taking the limit of a Riemann sum.

4.2.1 Sigma Notation

In the preceding section, you studied antidifferentiation. In this section, you will look further into a problem introduced in Section 1.1—that of finding the area of a region in the plane. At first glance, these two ideas may seem unrelated, but you will discover in Section 4.4 that they are closely related by an extremely important theorem called the Fundamental Theorem of Calculus.

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation**, a_i is the i**th** term of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 4.9:

1. $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

2. $\sum_{i=0}^{5}(i+1) = 1+2+3+4+5+6$

3. $\sum_{j=3}^{7} j^2 = 3^2+4^2+5^2+6^2+7^2$

4. $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^1+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

5. $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Notice that the first two are the same sum, just expressed in two different ways.

Although any variable can be used as the index of summation i, j, and k are the most common. Notice in Example 1 that the index of summation does not appear in the terms of the expanded sum.

The following properties of summation can be derived using the associative and commutative properties of addition and the distributive property of addition over multiplication. (In the first property, k is a constant.)

Properties of Summation

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

The next theorem lists some useful formulas for sums of powers. A proof of this theorem is given in Appendix A.

Theorem 4.2: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 4.10: Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000, 10000$.

Begin by applying Theorem 4.2 and write

$$\begin{aligned}
\sum_{i=1}^{n} \frac{i+1}{n^2} &= \frac{1}{n^2} \sum_{i=1}^{n} (i+1) \\
&= \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right) \\
&= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + n \right] \\
&= \frac{1}{n^2} \left[\frac{n^2+3n}{2} \right] \\
&= \frac{n+3}{2n}
\end{aligned}$$

And evaluate for each value of n.

Table 1

n	$\sum_{i=1}^{n} \frac{i+1}{n^2}$
10	0.65000
100	0.51500
1000	0.50150
10000	0.50015

In the table, note that the sum appears to approach a limit as n increases. Although the discussion of limits at infinity in Section 3.5 applies to a variable x, where x can be any real number, many of the same results hold true for limits involving the variable n, where n is restricted to positive integer values. So, we can find the limit

$$\lim_{n\to\infty} \frac{n+3}{2n} = \frac{1}{2}$$

Practice Exercises

Evaluate the sums and limits

13. Evaluate the sum: $\displaystyle\sum_{k=1}^{5}(2k-3)$

14. Use the summation formulas to evaluate: $\displaystyle\sum_{i=1}^{20}(2i^2-3)$

15. Write the sum in sigma notation: $\displaystyle\frac{2}{n}\left(1+\frac{2}{n}\right)^2+\frac{2}{n}\left(1+\frac{4}{n}\right)^2+\cdots+\frac{2}{n}\left(1+\frac{2n}{n}\right)^2$

16. Simplify the summation formula for n terms: $\displaystyle\sum_{i=1}^{n}\frac{3i}{n^2}$

17. Find the limit of the sum as $n \to \infty$: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n^2} + \frac{5}{n} \right)$

18. Given that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, evaluate $\lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$.

4.2.2 Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the *definition* of the **area of a rectangle**.

From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle; $A = \frac{1}{2}bh$. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown below.

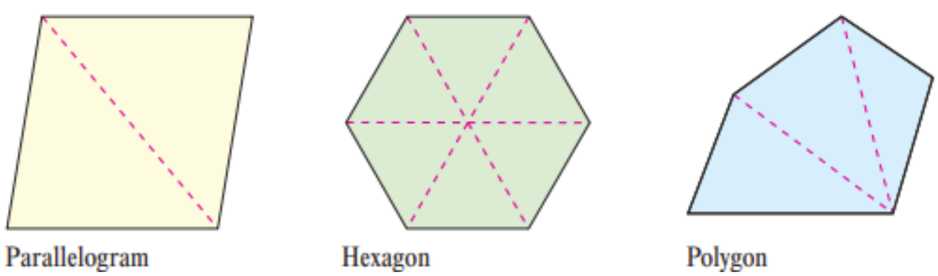


Figure 4.1

Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion method*. The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons-one inscribed in the region and one circumscribed about the region.

For instance, in the figure below, the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon. For each value of n the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle. Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.

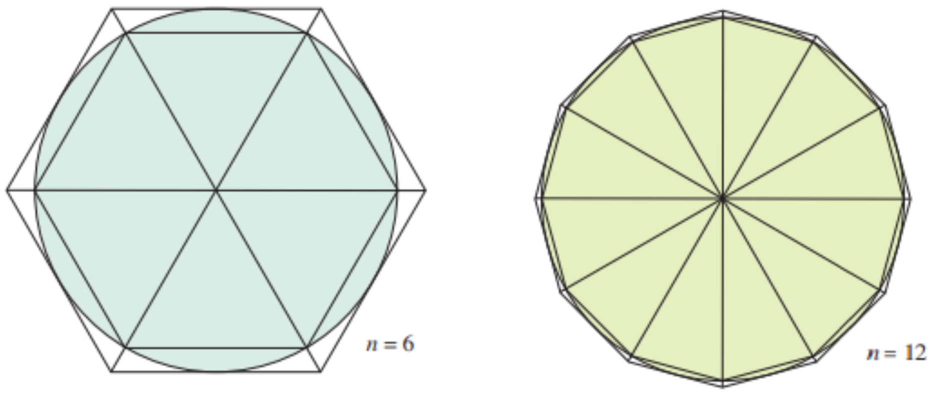


Figure 4.2 The exhaustion method for finding the area of a circular region.

A process that is similar to that used by Archimedes to determine the area of a plane region is used in the remaining examples in this section.

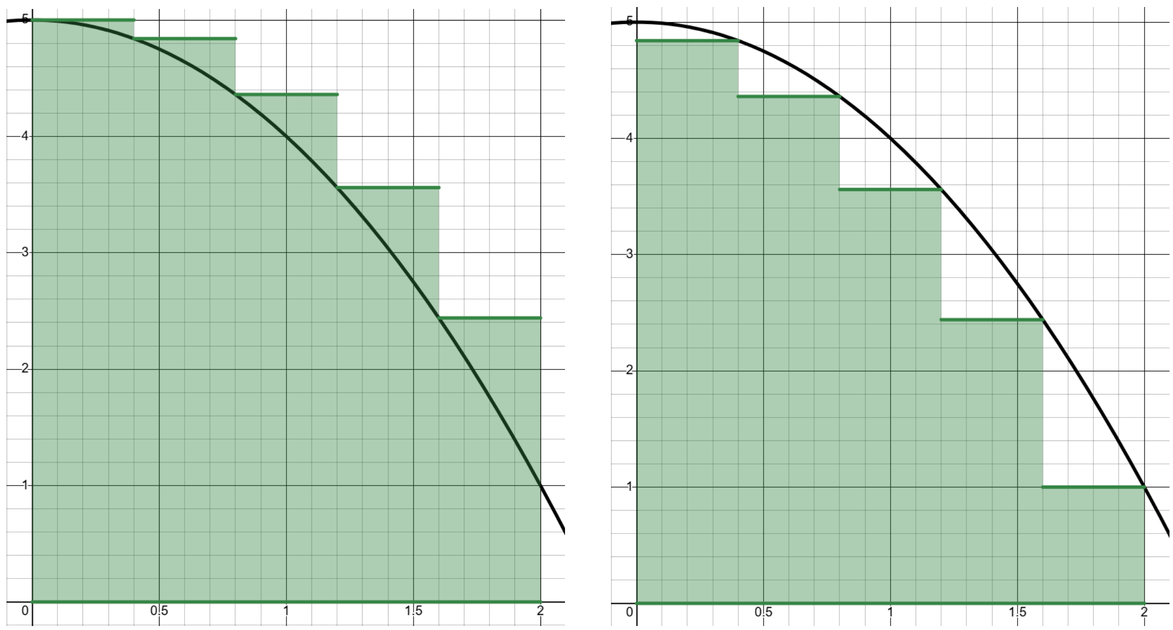
4.2.3 The Area of a Plane Region

Recall from Section 1.1 that the origins of calculus are connected to two classic problems: the tangent line problem and the area problem. The example below begins the investigation of the area problem.

Example 4.11: Use five rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.



The left endpoints (shown on the left) of the five intervals are $\frac{2}{5}(i-1)$ where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval.
The sum of the five rectangles is

$$\sum_{i=1}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08$$

Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.
The right endpoints (shown on the right) of the five intervals are $\frac{2}{5}i$ where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$ and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

The sum of the five rectangles is

$$\sum_{i=1}^{5} f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

Thus, you can conclude that

$$6.48 < (\text{Area of Region}) < 8.08$$

Notice that if we increased the number of rectangles, we could get a better approximation for the area. For example, increasing the number of rectangles from 5 to 25, we get

$$7.17 < (\text{Area of Region}) < 7.49$$

4.2.4 Upper and Lower Sums

The procedure in the previous example can be generalizes as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$. The region is bounded below by the x-axis, and the left and right boundaries of the region are vertical lines $x = a$ and $x = b$.

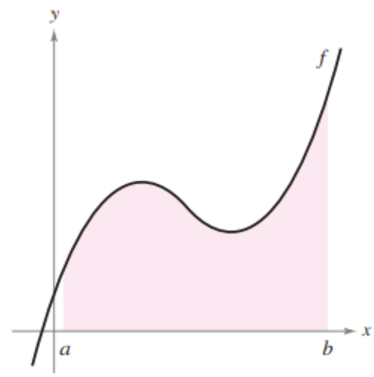


Figure 4.3 The region under a curve

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$.
The endpoints of each intervals are as follows:

$$\underbrace{a + 0(\Delta)x}_{a=x_0} < \underbrace{a + 1(\Delta)x}_{x_1} < \underbrace{a + 2(\Delta)x}_{x_2} < \cdots < \underbrace{a + n(\Delta)x}_{x_n=b}$$

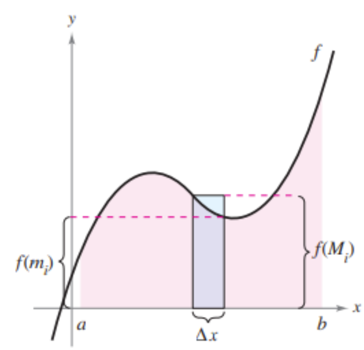


Figure 4.4 Interval $[a, b]$ divided into n subintervals

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$$f(m_i) = \text{Minimum value of } f(x) \text{ in } i\text{th subinterval}$$
$$f(M_i) = \text{Maximum value of } f(x) \text{ in } i\text{th subinterval}$$

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \leq f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of circumscribed rectangles is called an **upper sum**.

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$$

In the figure below, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between the two sums.

$$s(n) \le (\text{Area of Region}) \le S(n)$$

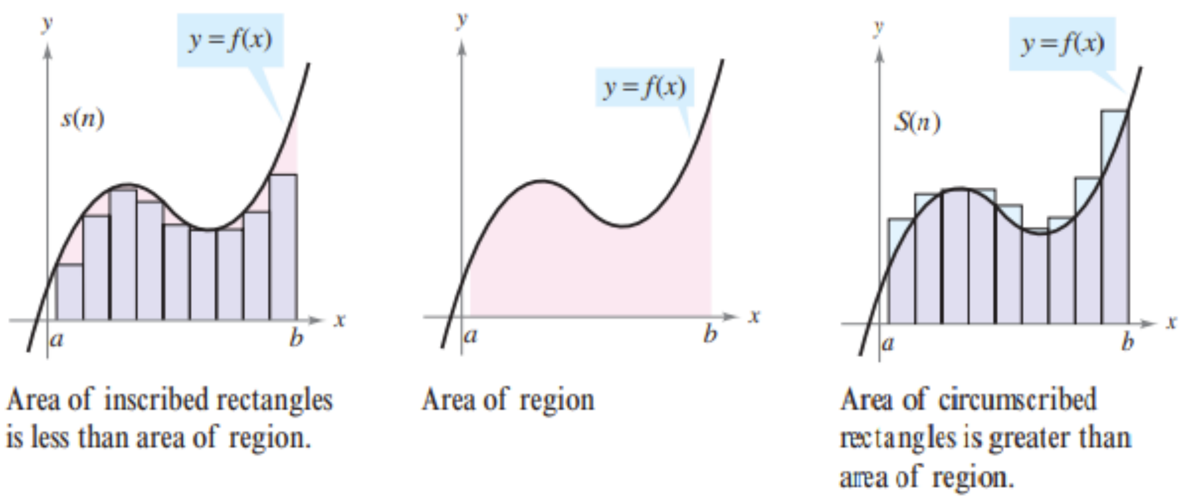


Figure 4.5

Example 4.12: Find the upper and lower sums for the region bounded by the graph $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.
To begin, partition the interval $[0, 2]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Because f is increasing on the interval $[0, 2]$, the minimum value of each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$\begin{aligned}
s(n) &= \sum_{i=1}^{n} f(m_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{2(i-1)}{n}\right)^2 \left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1) \\
&= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\
&= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right) \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

Using the right endpoints, the upper sum is

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} f(M_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right) i^2 \\
&= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{4}{3n^3}(2n^3 + 3n^2 + n) \\
&= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n. the lower sum is less than (or equal to) the upper sum.

$$s(n) == \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limits as $n \to \infty$, both the upper sum and lower sum converge to $\frac{8}{3}$

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Lower sum limit}$$

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Upper sum limit}$$

The next theorem shows that the equivalence of the limit (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$. The proof of this theorem is best left to a course in advanced calculus.

Theorem 4.3: Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x\\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem that the choice of x in the ith subinterval does not affect the limit. This means that the you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane.*

Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$

Example 4.13: Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.
Begin by noting that f is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$. According to the definition of area, you can choose any x-value in the ith subinterval. For this example, the right endpoints $c_i = i/n$ are convenient.

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \\
&= \frac{1}{4}
\end{aligned}$$

4.3 Riemann Sums and Definite Integrals

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Understand the definition of a Riemann sum.	I can explain what a Riemann sum represents and how it is constructed from partitions and sample points.
Evaluate a definite integral using limits.	I can evaluate a definite integral by writing it as a limit of Riemann sums.
Evaluate a definite integral using properties of definite integrals.	I can use properties of definite integrals to simplify and evaluate integrals without using limits.

4.3.1 Riemann Sum

I would like to note that I vary somewhat heavily from the textbook in this section. While I believe that the textbook does a really good job *mathematically*, I think my approach is more intuitive.
With that said, I maintain the same definitions and theorems presented in the book, if you choose to reference them.

The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In the preceding section, the function f was assumed to be continuous and nonnegative because we were dealing with area under a curve).

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

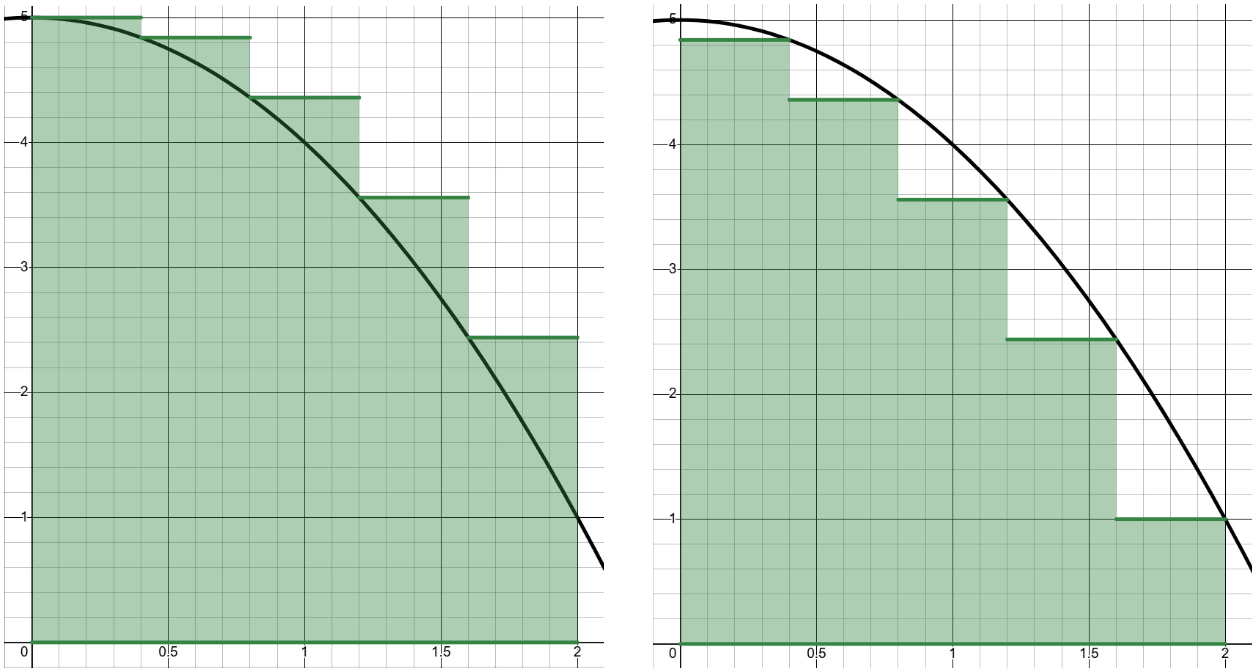
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

We previously explored Riemann Sums in 4.2.3. The example was $f(x) = -x^2 + 5$



In this example, we explored **Left-End Riemann Sums** and **Right-End Riemann Sums** to find that their sums were

$$\sum_{i=1}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08$$

and

$$\sum_{i=1}^{5} f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48$$

Example 4.14: Calculate the left and right Riemann sum of the function $f(x) = x^2 + x$ using 5 subintervals on the interval $[0, 3]$.
Begin by calculating the width of each subinterval:

$$\Delta x = \frac{3-0}{5} = \frac{3}{5}$$

Thus,

$$x_0 = 0, x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, x_3 = \frac{9}{5}, x_4 = \frac{12}{5}, x_5 = 3$$

Notice that there are 6 numbers. But be careful!

Now, for the left sum:

$$\begin{aligned}
L &= \sum_{i=0}^{4} f(x_i)\Delta x \\
&= f(x_0)\frac{3}{5} + f(x_1)\frac{3}{5} + f(x_2)\frac{3}{5} + f(x_3)\frac{3}{5} + f(x_4)\frac{3}{5} \\
&= 0\left(\frac{3}{5}\right) + \frac{24}{25}\left(\frac{3}{5}\right) + \frac{66}{25}\left(\frac{3}{5}\right) + \frac{126}{25}\left(\frac{3}{5}\right) + \frac{204}{25}\left(\frac{3}{5}\right) \\
&= \frac{3}{5}\left(0 + \frac{24}{25} + \frac{66}{25} + \frac{126}{25} + \frac{204}{25}\right) \\
&= \frac{3}{5}\left(\frac{420}{25}\right) \\
&= \frac{1260}{125} = \frac{252}{25} \\
&= 10.08
\end{aligned}$$

Now, for the right sum:

$$\begin{aligned}
R &= \sum_{i=1}^{5} f(x_i)\Delta x \\
&= f(x_1)\frac{3}{5} + f(x_2)\frac{3}{5} + f(x_3)\frac{3}{5} + f(x_4)\frac{3}{5} + f(x_5)\frac{3}{5} \\
&= \frac{24}{25}\left(\frac{3}{5}\right) + \frac{66}{25}\left(\frac{3}{5}\right) + \frac{126}{25}\left(\frac{3}{5}\right) + \frac{204}{25}\left(\frac{3}{5}\right) + 12\left(\frac{3}{5}\right) \\
&= \frac{3}{5}\left(\frac{24}{25} + \frac{66}{25} + \frac{126}{25} + \frac{204}{25} + \frac{300}{25}\right) \\
&= \frac{3}{5}\left(\frac{720}{25}\right) \\
&= \frac{2160}{125} = \frac{432}{25} \\
&= 17.28
\end{aligned}$$

Notice that the key difference between the left and the right Riemann sums comes from the selection of the endpoints.
You can visualize the differences easily using desmos.

Practice Exercises

19. Let $f(x) = x^2 + 2$. Use a right Riemann sum with $n = 4$ equal subintervals to approximate the area of the region bounded by the graph of f and the x-axis on the interval $[0, 2]$.

20. The function f is continuous and strictly increasing on the interval $[1, 5]$. A table of selected values is given below.

x	1	2	3	4	5
$f(x)$	2	4	7	11	18

Using 4 subintervals of equal length, determine if the Left Riemann Sum is an overestimate or an underestimate of the actual area under the curve. Justify your answer.

21. Use a Midpoint Riemann sum with $n = 3$ equal subintervals to approximate the area under the curve $f(x) = \frac{1}{x}$ on the interval $[1, 7]$.

22. The function $f(x) = \ln(x)$ is continuous on the interval $[2, 10]$.

1. Write the summation notation for the right Riemann sum approximation of the area under the graph of $f(x)$ on $[2, 10]$ using n subintervals of equal width.

2. Calculate the approximation using $n = 4$ subintervals.

23. Consider the region bounded by $f(x) = \sqrt{x}$ and the x-axis on the interval $[0, 9]$. Set up, but do not evaluate, the expression for the Right Riemann sum using n equal subintervals.

4.3.2 Definite Integrals

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $||\Delta||$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$||\Delta|| = \Delta x = \frac{b-a}{n} \quad \text{(Regular Partition)}$$

For a general partition, the norm is related to the number of subintervals of $[a, b]$ by

$$\frac{b-a}{||\Delta||} \leq n \quad \text{(General Partition)}$$

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $||\Delta|| \to 0$ implies that $n \to \infty$.

Note that the converse of this is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdots < \frac{1}{4} < \frac{1}{2} < 1$$

So, letting n approach infinity does not force $||\Delta||$ to approach 0. In a regular partition, however, the statements $||\Delta|| \to 0$ and $n \to \infty$ are equivalent.

To define the definite integral, consider the following limit.

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = L$$

To say that this limit exists means that for $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition with $||\Delta|| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$$

This must be true for any choice of c_i in the ith subinterval of Δ.

Definition of a Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

exists, then f is **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

The limit is called the **definite integral** of f from a to b.

- a is the **lower limit** of integration.
- b is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now it is important to see that definite integrals and indefinite integrals are different identities. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

A sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this class.

Theorem 4.4: Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

Theorem 4.5: The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)dx$$

While the previous theorem states that continuity implies integrability, we must refine our geometric interpretation of this integral. In strict geometric terms, "area" is always a positive quantity. However, for the purposes of the definite integral and the AP Calculus exam, we utilize the concept of **net signed area**.

Theorem: The Definite Integral as Net Signed Area

If f is continuous on the closed interval $[a, b]$, the definite integral represents the **net signed area** bounded by the graph of f and the x-axis.
If we let A_{up} be the area of the geometric region between the graph of f and the x-axis where $f(x) \geq 0$, and A_{down} be the area of the geometric region where $f(x) < 0$ (as shown in Figure 4.6), then:

$$\int_a^b f(x)\,dx = A_{up} - A_{down}$$

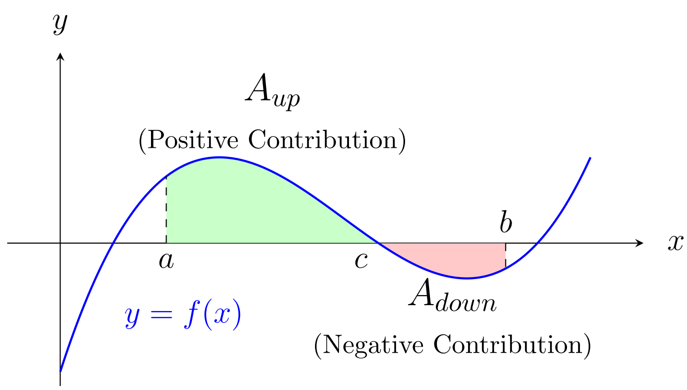


Figure 4.6 Visualizing Net Signed Area. A_{up} and A_{down} represent positive geometric areas. The integral counts A_{up} positively and A_{down} negatively.

It is vital to distinguish between the integral of a function and the total geometric area involved.

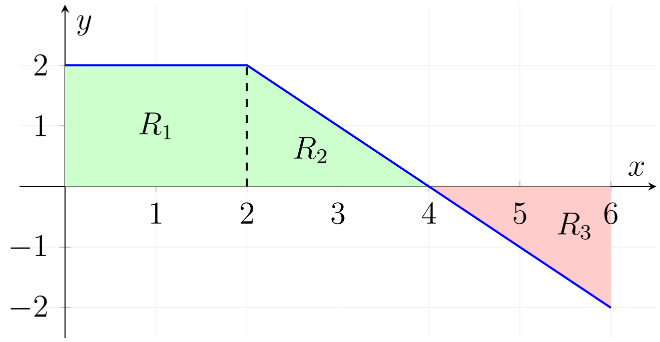
- **Net Signed Area (Displacement):** This is the value of the definite integral. Regions below the x-axis subtract from the total.

$$\text{Net Area} = \int_a^b f(x)\,dx = A_{up} - A_{down}$$

- **Total Geometric Area (Distance):** This is the sum of the absolute areas of all regions, treating negative regions as positive. This is calculated by integrating the absolute value of the function.

$$\text{Total Area} = \int_a^b |f(x)|\,dx = A_{up} + A_{down}$$

Example 4.15: Consider the function f defined by the graph below. Evaluate the definite integral $\int_0^6 f(x)\,dx$ and calculate the total area of the region between the graph and the x-axis.



Begin by identifying the geometric shapes formed by the graph and the x-axis.

- Region R_1 (on $[0, 2]$) is a rectangle with width 2 and height 2.
- Region R_2 (on $[2, 4]$) is a triangle with base 2 and height 2.
- Region R_3 (on $[4, 6]$) is a triangle with base 2 and height 2 (below the axis).

First, calculate the geometric area of each shape:

$$\begin{aligned}
\text{Area}(R_1) &= \text{width} \cdot \text{height} = 2 \cdot 2 = 4 \\
\text{Area}(R_2) &= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot 2 = 2 \\
\text{Area}(R_3) &= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot 2 = 2
\end{aligned}$$

Notice that geometric area is always positive, even for the region below the axis!

Now, to find the **Definite Integral** (Net Signed Area), we treat regions above the x-axis as positive and regions below as negative:

$$\begin{aligned}
\int_0^6 f(x)\,dx &= (\text{Area } R_1) + (\text{Area } R_2) - (\text{Area } R_3) \\
&= 4 + 2 - 2 \\
&= 4
\end{aligned}$$

Finally, to find the **Total Area** (Total Distance), we sum the absolute values of all areas:

$$\begin{aligned}
\text{Total Area} &= \int_0^6 |f(x)|\,dx \\
&= |\text{Area } R_1| + |\text{Area } R_2| + |\text{Area } R_3| \\
&= 4 + 2 + 2 \\
&= 8
\end{aligned}$$

You can check this result by noticing that the positive triangle R_2 and the negative triangle R_3 have equal areas and cancel each other out in the definite integral, leaving only the area of the rectangle R_1.

Practice Exercises

Evaluate the integrals and answer the conceptual questions

24. Converting Limits to Integrals: Which of the following definite integrals is equivalent to the limit given below?

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{3i}{n}\right)^2 \left(\frac{3}{n}\right)$$

1. $\displaystyle\int_4^7 x^2\,dx$
2. $\displaystyle\int_0^3 (4+x)^2\,dx$
3. $\displaystyle\int_0^3 x^2\,dx$
4. Both (a) and (b)

25. Geometric Evaluation: Sketch the region corresponding to the definite integral and evaluate it using a geometric formula.

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

26. Net Signed Area: Evaluate the definite integral $\int_0^6 (x-2)\,dx$ by interpreting it in terms of areas. Explain why the answer is positive or negative.

27. Total Area vs. Net Area: Consider the function $f(x) = 2x - 6$ on the interval $[0, 5]$.

1. Evaluate the definite integral (Net Signed Area): $\int_0^5 (2x-6)\,dx$
2. Evaluate the total geometric area (Total Distance): $\int_0^5 |2x-6|\,dx$

28. Graphing and Integration: The graph of the function f consists of a semicircle of radius 2 centered at $(2, 0)$ and a line segment from $(4, 0)$ to $(6, 2)$. Evaluate $\int_0^6 f(x)\,dx$.

29. Particle Motion Application: A particle moves along the x-axis with velocity $v(t) = 4 - 2t$ for $0 \leq t \leq 4$.

1. Find the displacement of the particle from $t = 0$ to $t = 4$.
2. Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

4.3.3 Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$. Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then we define

$$\int_a^a f(x)dx = 0$$

2. If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Example 4.16:

1. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_\pi^\pi \sin(x)dx = 0$$

2. Consider the integral $\int_3^0 (x+2)dx$. By the second definition, we have

$$\int_3^0 (x+2)dx = -\int_0^3 (x+2)dx = \frac{21}{2} = 10.5$$

   Because we can find the area of a right trapezoid geometrically.

Theorem 4.6: Additive Interval Property

If f is integrable on three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

We actually used this theorem earlier without explicitly describing it. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation (the properties were given in section 4.2.1 Sigma Notation).

Theorem 4.7: Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is constant, then the functions of kf and $f \pm g$ are integrable on $[a, , b]$, and

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (Constant Multiple Rule)
2. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ (Sum and Difference Rule)

Note that the second property can be extended to cover any finite number of functions. That is, the integrand could be $f(x) + g(x) + \cdots + h(x)$

Example 4.17: If $\int_0^5 f(x)dx = 10$ and $\int_0^5 g(x)dx = 3$, find $\int_0^5 [3f(x) - g(x)]dx$.

$$\begin{aligned}
\int_0^5 [3f(x) - g(x)]\,dx &= \int_0^5 3f(x)\,dx - \int_0^5 g(x)\,dx && \text{Sum and Difference Rule} \\
&= 3\int_0^5 f(x)\,dx - \int_0^5 g(x)\,dx && \text{Constant Multiple Rule} \\
&= 3(10) - (3) && \text{Substitute given values} \\
&= 30 - 3 \\
&= 27
\end{aligned}$$

Theorem 4.8: Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then
$$0 \leq \int_a^b f(x)dx$$
2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Example 4.18: Without evaluating the integral, explain why $\int_0^1 \sqrt{1+x^2}\,dx \geq \int_0^1 x\,dx$.

To show this without evaluating, we must compare the integrands (the functions inside the integrals) on the interval $[0,1]$.
For all x in the interval $[0,1]$:

$1+x^2 \geq x^2$	Since $1>0$, adding it makes the value larger.
$\sqrt{1+x^2} \geq \sqrt{x^2}$	Taking the square root preserves the inequality.
$\sqrt{1+x^2} \geq \lvert x\rvert$	Definition of $\sqrt{x^2}$.
$\sqrt{1+x^2} \geq x$	Since $x \geq 0$ on the interval $[0,1]$.

Since the function $f(x)=\sqrt{1+x^2}$ is always greater than or equal to $g(x)=x$ on the interval, the **Preservation of Inequality Theorem** guarantees that the area under $f(x)$ is greater than or equal to the area under $g(x)$. Therefore:

$$\int_0^1 \sqrt{1+x^2}\,dx \geq \int_0^1 x\,dx$$

Practice Exercises

Use the properties of definite integrals to solve

30. Given $\displaystyle\int_0^5 f(x)\,dx = 10$ and $\displaystyle\int_0^5 g(x)\,dx = -3$, evaluate $\displaystyle\int_0^5 [2f(x)-4g(x)]\,dx$.

31. Given $\displaystyle\int_{-2}^2 h(x)\,dx = 4$ and $\displaystyle\int_2^5 h(x)\,dx = 3$, find the value of $\displaystyle\int_{-2}^5 h(x)\,dx$.

32. Given $\int_1^7 f(x)\,dx = 15$ and $\int_1^4 f(x)\,dx = 6$, find the value of $\int_4^7 f(x)\,dx$.

33. Write the following expression as a single definite integral of the form $\int_a^b f(x)\,dx$:

$$\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx + \int_6^2 f(x)\,dx$$

34. If $\int_2^6 f(x)\,dx = 8$, evaluate $\int_2^6 (f(x) + 5)\,dx$.
(Hint: Recall that $\int_a^b k\,dx$ represents the area of a rectangle with height k and width $b - a$).

35. Determine the value of the integral $\int_5^5 (\sin(x^2) + e^x)\,dx$.

4.4 The Fundamental Theorem of Calculus

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Evaluate a definite integral using the Fundamental Theorem of Calculus.	I can evaluate a definite integral by finding an antiderivative and applying the Fundamental Theorem of Calculus.
Understand and use the Mean Value Theorem for Integrals.	I can apply the Mean Value Theorem for Integrals to find an average value guaranteed on an interval.
Find the average value of a function over a closed interval.	I can compute the average value of a function on a closed interval using a definite integral.
Understand and use the Second Fundamental Theorem of Calculus.	I can differentiate an accumulation function using the Second Fundamental Theorem of Calculus.

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus**.

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in the figure below. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.

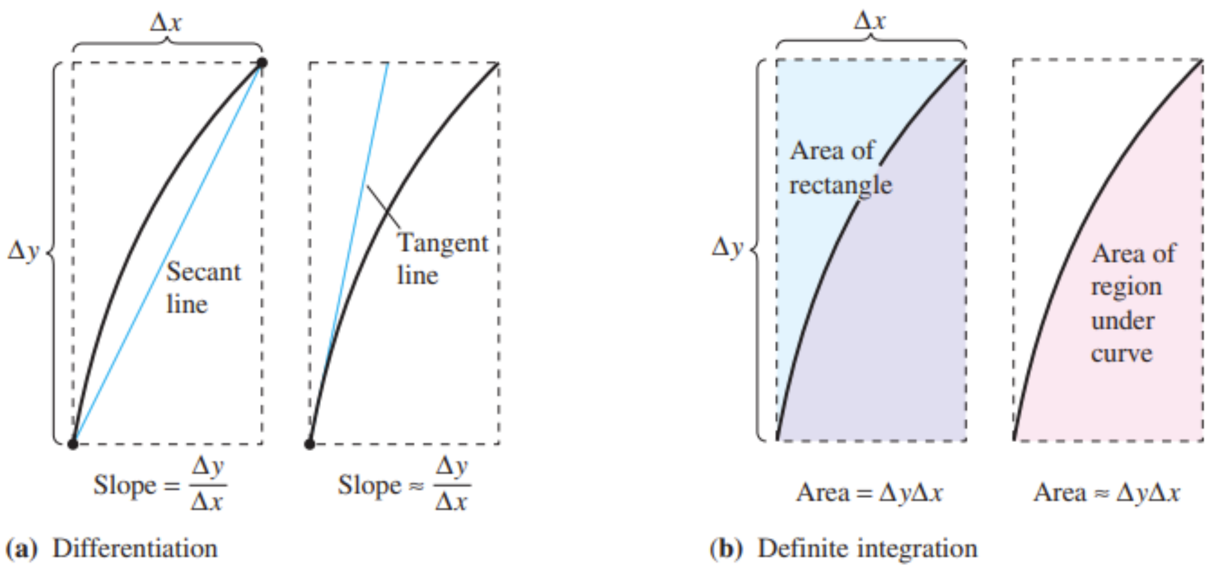


Figure 4.7 Differentiation and definite integration have an "inverse" relationship.

> **Theorem 4.9**: The Fundamental Theorem of Calculus
>
> If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$

Proof: The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be the following partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This important equation tells you that by applying the Mean Value Theorem you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$. Taking the limit (as $||\Delta|| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)dx$$

□

Guidelines for Using the Fundamental Theorem of Calculus

1. Provided that you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus the following notation, an evaluation bar, is convenient.

$$\begin{aligned} \int_a^b f(x)dx &= F(x)\Big|_a^b \\ &= F(b) - F(a) \end{aligned}$$

   For example,

$$\begin{aligned} \int_1^3 x^3 dx &= \frac{x^4}{4}\Big|_1^3 \\ &= \frac{3^4}{4} - \frac{1^4}{4} \\ &= \frac{81}{4} - \frac{1}{4} \\ &= 20 \end{aligned}$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned} \int_a^b f(x)dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

   Notice that you can also use brackets as an evaluation bar.

Example 4.19: Evaluate each definite integral.

1. $\displaystyle\int_1^2 (x^2-3)dx$

$$\int_1^2 (x^2-3)dx = \left[\frac{x^3}{3}-3x\right]_1^2 = \left(\frac{8}{3}-6\right)-\left(\frac{1}{3}-3\right) = -\frac{2}{3}$$

2. $\displaystyle\int_1^4 3\sqrt{x}dx$

$$\int_1^4 3\sqrt{x}dx = 3\int_1^4 \sqrt{x}dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2}-2(1)^{3/2} = 14$$

3. $\displaystyle\int_0^{\pi/4} \sec^2(x)dx$

$$\int_0^{\pi/4} \sec^2(x)dx = \tan(x)\Big|_0^{\pi/4} = 1-0 = 1$$

Example 4.20: Evaluate $\displaystyle\int_0^2 |2x-1|dx$.

We can rewrite the integrand by using the definition of absolute value:

$$|2x-1| = \begin{cases} -(2x-1), & x<\frac{1}{2} \\ 2x-1, & x\geq\frac{1}{2}\end{cases}$$

Now, you can rewrite the integral in two parts

$$\begin{aligned}\int_0^2 |2x-1|dx &= \int_0^{1/2} -(2x-1)dx + \int_{1/2}^2 (2x-1)dx \\ &= \Big[-x^2+x\Big]_0^{1/2} + \Big[x^2-x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4}+\frac{1}{2}\right)-(0-0)+(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$

Practice Exercises

Evaluate the following definite integrals

36. $\int_{-1}^{2} (3x^2 - 2x + 1)\, dx$

37. $\int_{1}^{8} \sqrt[3]{x}\, dx$

38. $\int_{0}^{\pi} (2 + \cos(x))\, dx$

39. $\int_{0}^{4} |x - 3|\, dx$

40. $\int_{1}^{2} \frac{3x^2 + 5}{x^2}\, dx$

41. Let $F(x)$ be an antiderivative of $f(x)$. If $\int_2^5 f(x)\,dx = 12$ and $F(5) = 18$, find the value of $F(2)$.

4.4.1 Mean Value theorem for Integrals

In Section 4.2.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve.

Theorem 4.10: Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Proof: **Case 1:** If f is constant on the interval $[a,b]$, the theorem is clearly valid because c can be any point in $[a,b]$.

Case 2: If f is not constant on $[a,b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a,b]$. Because $f(m) \le f(x) \le f(M)$ for all x in $[a,b]$, you can apply Theorem 4.8 to write the following.

$$\begin{aligned}
\int_a^b f(m)dx &\le \int_a^b f(x)dx \le \int_a^b f(M)dx \\
f(m)(b-a) &\le \int_a^b f(x)dx \le f(M)(b-a) \\
f(m) &\le \frac{1}{b-a}\int_a^b f(x)dx \le f(M)
\end{aligned}$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \quad \text{or} \quad f(c)(b-a) = \int_a^b f(x)dx$$

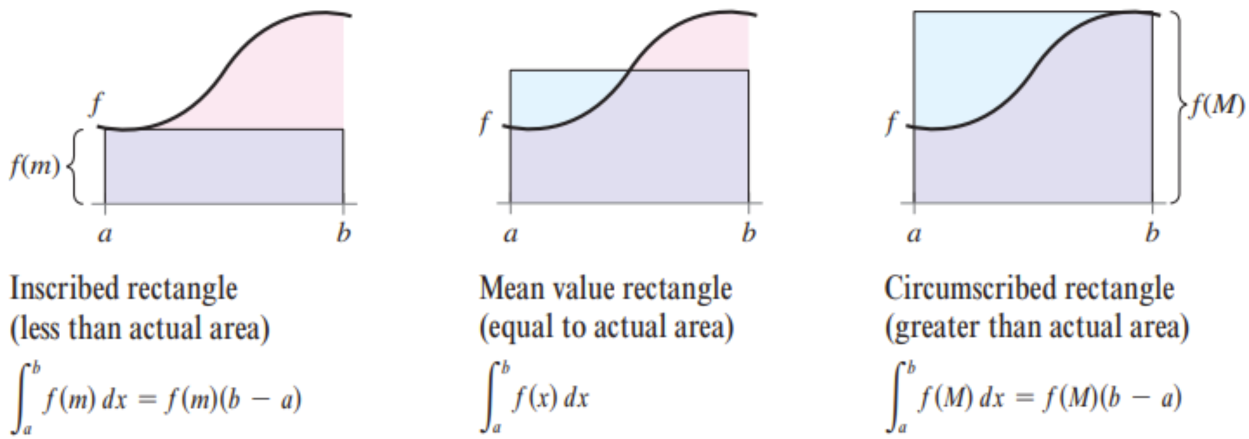


Figure 4.8

Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval.

□

4.4.2 Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is:

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)dx$$

To see why the average value of f is defined this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \cdots + f(c_n)\right]$$

By multiplying and dividing by $(b-a)$, you can write the average as

$$\begin{aligned} a_n = \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 7, you will study other applications, such as volume, arc length, centers of mass, and work.

Example 4.21: Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.
Begin by using the definition of average value

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{4-1}\int_1^4 (3x^2 - 2x)dx \\ &= \frac{1}{3}\left[x^3 - x^2\right]_1^4 \\ &= \frac{1}{3}[64 - 16 - (1 - 1)] \\ &= \frac{48}{3} \\ &= 16\end{aligned}$$

Practice Exercises

Find the average value and apply the Mean Value Theorem

42. Find the average value of the function $f(x) = x^3 - x$ on the interval $[0, 2]$.

43. Find the average value of $g(x) = \sin(x)$ on the interval $[0, \pi]$.

44. Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = 4 - x^2$ on the interval $[0, 2]$.

45. The temperature (in °F) in a room from 9 AM to 2 PM is modeled by the function $T(t) = 70 + 2\sqrt{t}$, where t is measured in hours starting at 9 AM ($t = 0$). Find the average temperature of the room during this 5-hour period.

46. Given that the average value of a continuous function f on the interval $[2, 6]$ is 12, evaluate the definite integral $\displaystyle\int_2^6 f(x)\,dx$.

47. Find the average value of $h(x) = e^{2x}$ on the interval $[0, \ln(3)]$.

4.4.3 The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used as the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration.

The Definite Integral as a Number

$$\int_a^b f(x)\,dx$$

- a is a constant
- b is a constant
- f is a function of x

The Definite Integral as a Function of x

$$F(x) = \int_a^x f(t)\,dt$$

- a is a constant
- F is a function of x
- f is a function of t

Example 4.22: Evaluate the function

$$F(x) = \int_0^x \cos(t)dt$$

at $x = 0, \pi/6, \pi/4, \pi/3, \pi/2$.
You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily to obtain

$$\int_0^x \cos(t)dt = \sin(x)\Big|_0^x = \sin(x) - \sin(0) = \sin(x)$$

Now, simply use $F(x) = \sin(x)$ to get

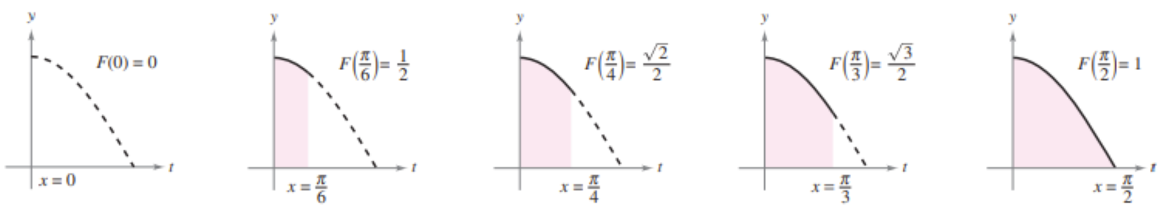


Figure 4.9 $F(x) = \int_0^x \cos(t)dt$ is the area under the curve $f(t) = \cos(t)$ from 0 to x.

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos(t)$ from $t = 0$ to $t = x$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

Notice that in the previous example, the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin(x)] = \frac{d}{dx}\left[\int_0^x \cos(t)dt\right] = \cos(x)$$

This result is generalized in the following theorem, the **Second Fundamental Theorem of Calculus**.

Theorem 4.11: The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Proof: Begin by defining F as

$$F(x) = \int_a^x f(t)dt$$

Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\int_a^{x+\Delta x} f(t)dt + \int_x^a f(t)dt \right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\int_x^{x+\Delta x} f(t)dt \right] \end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x+\Delta x]$ such that the integral in the expression above is equal to $f(c)\Delta x$. Moreover, because $x \leq c \leq c + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} f(c) \Delta x \right] \\ &= \lim_{\Delta x \to 0} f(c) \\ &= f(x) \end{aligned}$$

A similar argument can be made for $\Delta x < 0$.

□

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function.

Example 4.23: Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}dt\right]$.

Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}dt\right] = \sqrt{x^2+1}$$

Applying the Chain Rule If the upper limit is not just x but a function $u(x)$, we must apply the Chain Rule.

$$\frac{d}{dx}\left[\int_a^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x)$$

Example 4.24: Find the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos(t)dt$.

Let $u = x^3$, and see that

$$\begin{aligned}
F'(x) &= \frac{dF}{du}\frac{du}{dx} && \text{Chain Rule} \\
&= \frac{d}{du}[F(x)]\frac{du}{dx} && \text{Definition of } \frac{dF}{du} \\
&= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos(t)dt\right]\frac{du}{dx} && \text{Substitute } \int_{\pi/2}^{x^3} \cos(t)dt \text{ for } F(x) \\
&= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos(t)dt\right]\frac{du}{dx} && \text{Substitute } u \text{ for } x^3 \\
&= (\cos(u))(3x^2) && \text{Apply Second Fundamental Theorem of Calculus} \\
&= (\cos(x^3))(3x^2) && \text{Rewrite as a function of } x
\end{aligned}$$

Practice Exercises

Apply the Second Fundamental Theorem of Calculus

48. Evaluate the derivative: $\frac{d}{dx}\left[\int_3^x \sqrt{t^2+4}\,dt\right]$.

49. Find $F'(x)$ given $F(x) = \int_1^{x^3} \frac{1}{t^2+1}\,dt$.

50. Find $\frac{dy}{dx}$ for the function $y = \int_x^5 \cos(t^2)\,dt$.
(Hint: Switch the limits of integration first.)

51. Let $g(x) = \int_0^{2x} f(t)\,dt$. If $f(t) = e^t + t$, find $g'(0)$.

52. Find the equation of the tangent line to the graph of $F(x) = \int_1^x \sqrt{t^3+8}\,dt$ at the point where $x = 2$.

53. Let $H(x) = \int_0^x f(t)\,dt$, where f is the continuous function defined by $f(t) = t^2 - 4$.

1. Find $H'(x)$.
2. Find $H''(x)$.
3. On what interval(s) is $H(x)$ concave up?

4.5 Integration By substitution

Lesson Objectives & Success Criteria

Key Topics & Formulas	Success Criteria
Use pattern recognition to find an indefinite integral.	I can recognize patterns that match known derivatives and use them to find indefinite integrals.
Use a change of variables to find an indefinite integral.	I can use a change of variables to rewrite and evaluate an indefinite integral.
Use the General Power Rule for Integration to find an indefinite integral.	I can apply the General Power Rule to find indefinite integrals involving powers of x.
Use a change of variables to evaluate a definite integral.	I can use a change of variables to evaluate a definite integral and correctly adjust the limits of integration.
Evaluate a definite integral involving an even or odd function.	I can use symmetry and properties of even and odd functions to simplify and evaluate definite integrals.

4.5.1 Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts-*pattern recognition* and *change of variables.* Both techniques result in a u**-substitution**. With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

From the definition of an antiderivative, it follows that

$$\begin{aligned}\int F'(g(x))g'(x)dx &= F(g(x)) + C\\ &= F(u) + C\end{aligned}$$

The results are summarized in the following theorem.

Theorem 4.12: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\text{Outside Function}}{\overset{\downarrow}{f}}\big(\underset{\underset{\text{Inside Function}}{\uparrow}}{g(x)}\big)\overbrace{g'(x)}^{\text{Derivative of Inside}}\,dx = F(g(x)) + C$$

Example 4.25: Find $\int (x^2+1)^2(2x)dx$.

Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2+1) = (x^2+1)^2$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.12, you can write

$$\int \underbrace{(x^2+1)^2}_{f(g(x))}\underbrace{(2x)}_{g'(x)}\,dx = \frac{1}{3}(x^2+1)^3 + C$$

Example 4.26: Find $\int 5\cos(5x)dx$.

Letting $g(x) = 5x$, you get

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos(5x)$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for Integration and Theorem 4.12, you can write

$$\int \underbrace{(\cos(5x))}_{f(g(x))} \underbrace{(5)}_{g'(x)} dx = \sin(5x) + C$$

You can check this by differentiating $\sin(5x) + C$ to obtain the original integrand.

The integrands in the previous examples fit the $f(g(x))g'(x)$ pattern exactly. You only had to recognize the pattern! You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)dx = k\int f(x)dx$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in the next example.

Example 4.27: Find $\int x(x^2+1)^2 dx$.

This is similar to the integral given in **Example 4.24**, except it is missing a factor of 2. Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let $g(x) = x^2 + 1$ and supply the $2x$ as follows.

$$\begin{aligned}
\int x(x^2+1)^2 dx &= \int (x^2+1)^2 \left(\frac{1}{2}\right)(2x)dx && \text{Multiply and divide by 2} \\
&= \frac{1}{2}\int \underbrace{(x^2+1)^2}_{f(g(x))} \underbrace{(2x)}_{g'(x)} dx && \text{Constant Multiple Rule} \\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C && \text{Integrate} \\
&= \frac{1}{6}(x^2+1)^3 + C && \text{Simplify}
\end{aligned}$$

4.5.2 Change of Variables

With a formal **change of variables**, you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated before, it is useful for complicated integrands. The change of variable technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)dx$, and the integral in Theorem 4.12 takes the form

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

Example 4.28: Find $\int \sqrt{2x-1}dx$.

First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = \frac{1}{2}du$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}dx &= \int \sqrt{u}\frac{1}{2}du && \text{Integral in terms of } u \\
&= \frac{1}{2}\int u^{1/2}du && \text{Constant Multiple Rule} \\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u \\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify} \\
&= \frac{1}{3}(2x-1)^{3/2} + C && \text{Antiderivative in terms of } x
\end{aligned}$$

Sometimes, simply replacing $g(x)$ with u leaves extra x's in the integrand. In these cases, you must solve the substitution equation for x to rewrite the entire integral in terms of u.

Example 4.29: Find $\int x\sqrt{2x-1}dx$.

Just like the previous example, let $u = 2x - 1$ and obtain $dx = \frac{1}{2}du$. Because the integrand contains a factor of x, we must also solve for x in terms of u.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2$$

$$\begin{aligned} \int x\sqrt{2x-1}dx &= \int \left(\frac{u+1}{2}\right) u^{1/2}\frac{1}{2}du \\ &= \frac{1}{4}\int (u^{3/2} + u^{1/2})du \\ &= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C \\ &= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C \end{aligned}$$

Example 4.30: Find $\int \sin^2(3x)\cos(3x)dx$.

Because $\sin^2(3x) = (\sin(3x))^2$ you can let $u = \sin(3x)$. Then

$$du = \cos(3x)(3)dx$$

Now, because $\cos(3x)dx$ is part of the original integral, you can write

$$\frac{1}{3}du = \cos(3x)dx$$

Substituting u and $\frac{1}{3}du$ in the original integral yields

$$\begin{aligned} \int \sin^2(3x)\cos(3x)dx &= \int u^2\frac{1}{3}du \\ &= \frac{1}{3}\int u^2 du \\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \\ &= \frac{1}{9}\sin^3(3x) + C \end{aligned}$$

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the **inner part** of a composite function.
2. Compute $du = g'(x)dx$.
3. Rewrite the integral **entirely** in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Practice Exercises

Find the indefinite integral

54. $\displaystyle\int (5x+1)^4\,dx$

55. $\displaystyle\int x(x^2-9)^3\,dx$

56. $\displaystyle\int \frac{x^2}{\sqrt{x^3+8}}\,dx$

57. $\displaystyle\int \cos^3(2x)\sin(2x)\,dx$

58. $\int x\sqrt{x-4}\,dx$

59. $\int \frac{x}{\sqrt{2x+1}}\,dx$

4.5.3 The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because the importance of this type of substitution, it is given a special name - the **General Power Rule for Integration**. A proof of this rule follows directly from the Power Rule for Integration, together with Theorem 4.12.

Theorem 4.13: The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq 1$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq 1$$

Example 4.31:

1.

$$\int 3(3x-1)^4\,dx = \int \underbrace{(3x-1)^4}_{u^4}\,\underbrace{(3)\,dx}_{du} = \underbrace{\frac{(3x-1)^5}{5}}_{u^5/5} + C$$

2.
$$\int (2x+1)(x^2+x)\,dx = \int \underbrace{(x^2+x)^1}_{u^1}\,\underbrace{(2x+1)\,dx}_{du} = \underbrace{\frac{(x^2+x)^2}{2}}_{u^2/2} + C$$

3.
$$\int 3x^2\sqrt{x^3-2}\,dx = \int \underbrace{(x^3-2)^{1/2}}_{u^{1/2}}\,\underbrace{(3x^2)\,dx}_{du} = \underbrace{\frac{(x^3-2)^{3/2}}{3/2}}_{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2}$$

4.
$$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \underbrace{(1-2x^2)^{-2}}_{u^{-2}}\,\underbrace{(-4x)\,dx}_{du} = \underbrace{\frac{(1-2x^2)^{-1}}{-1}}_{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

5.
$$\int \cos^2 x \sin x\,dx = -\int \underbrace{(\cos x)^2}_{u^2}\,\underbrace{(-\sin x)\,dx}_{du} = \underbrace{-\frac{(\cos x)^3}{3}}_{u^3/3} + C$$

4.5.4 Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivatives back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.12 combined with the Fundamental Theorem of Calculus.

Theorem 4.14: Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 4.32: Evaluate $\int_0^1 x(x^2+1)^3 dx$.
To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \quad \Rightarrow \quad du = 2x dx$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit: When $x = 0$, $u = 0^2 + 1 = 1$
Upper Limit: When $x = 1$, $u = 1^2 + 1 = 2$

Now, you can substitute to obtain

$$\begin{aligned}
\int_0^1 x(x^2+1)^3 dx &= \frac{1}{2}\int_0^1 (x^2+1)(2x)dx && \text{Multiply and divide by 2} \\
&= \frac{1}{2}\int_1^2 u^3 du && \text{Notice the new limits of integration} \\
&= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 \\
&= \frac{1}{2}\left(4 - \frac{1}{4}\right) \\
&= \frac{15}{8}
\end{aligned}$$

Notice that if we instead rewrote the antiderivative in terms of the variable x and used the original limits of integration, we would get the same answer:

$$\begin{aligned}
\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 &= \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1 \\
&= \frac{1}{2}\left(4 - \frac{1}{4}\right) \\
&= \frac{15}{8}
\end{aligned}$$

Example 4.33: Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}dx$.
To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

$$\begin{aligned} u^2 &= 2x - 1 \\ u^2 + 1 &= 2x \\ \frac{u^2+1}{2} &= x \\ udu &= dx \end{aligned}$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit: When $x = 1, \quad u = \sqrt{2-1} = 1$
Upper Limit: When $x = 5, \quad u = \sqrt{10-1} = 3$

Now, substitute to obtain

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}}dx &= \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right)udu \\ &= \frac{1}{2}\int_1^3 (u^2+1)du \\ &= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) \\ &= \frac{16}{3} \end{aligned}$$

Geometrically, you can interpret

$$\int_1^5 \frac{x}{\sqrt{2x-1}}dx = \int_1^3 \left(\frac{u^2+1}{2}\right)du$$

to mean that the two *different* regions have the *same* area.

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual.

For example, after substituting $u = \sqrt{1-x}$ in the integral

$$\int_0^1 x^2(1-x)^{1/2}dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

$$-2\int_1^0 (1-u^2)^2 u^2 du$$

Practice Exercises

Evaluate the definite integrals using the method of change of variables

60. $\int_0^2 3x^2(x^3+1)^3\,dx$

61. $\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx$

62. $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

63. $\int_0^1 x\sqrt{1-x}\,dx$

64. Rewrite the integral $\int_0^2 e^{2x}(1+e^{2x})^5\,dx$ in terms of u, where $u = 1+e^{2x}$. Determine the new limits of integration, but do not evaluate the final result.

65. $\int_1^e \frac{\ln(x)}{x}\,dx$

4.5.5 Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral (over an interval that is symmetric about the y-axis or about the origin) by recognizing the integrand to be an even or odd function.

Definition of Even and Odd Functions

The function $y = f(x)$ is **even** if $f(-x) = f(x)$. (Symmetric about the y-axis)
The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$. (Symmetric about the origin)

Theorem 4.15: Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an **even** function, then:

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

2. If f is an **odd** function, then:

$$\int_{-a}^{a} f(x)dx = 0$$

Proof: Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.12 with the substitution $u = -x$ (so $du = -dx$, and the limits flip from $x = -a \to u = a$ and $x = 0 \to u = 0$) produces

$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-u)(-1)du = -\int_{a}^{0} f(u)du = \int_{0}^{a} f(u)du$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\ &= \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx \\ &= 2\int_{0}^{a} f(x)dx\end{aligned}$$

Thus proving the first property.
The proof of the second property is left to you.

$\square$

Example 4.34: Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \big((\sin^3(x)\cos(x) + \sin(x)\cos(x)\big)\, dx$.
Letting $f(x) = \sin^3(x)\cos(x) + \sin(x)\cos(x)$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x) \\ &= -\sin^3(x)\cos(x) - \sin(x)\cos(x) \\ &= -f(x)\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$ you can apply Theorem 4.15 to conclude that

$$\int_{-\pi/2}^{\pi/2} \big((\sin^3(x)\cos(x) + \sin(x)\cos(x)\big)\, dx = 0$$

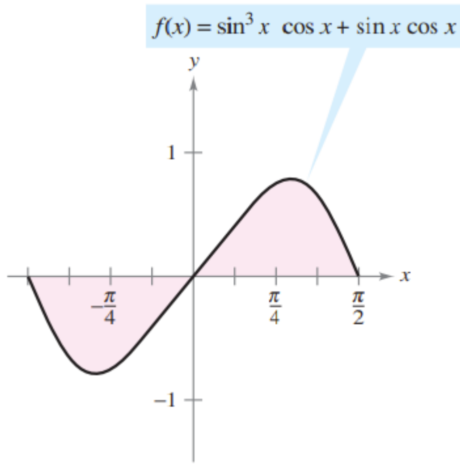


Figure 4.10 Because f is an odd function, $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$

4.6 Numerical Integration

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Approximate a definite integral using the Midpoint Rule.	I can approximate a definite integral using the Midpoint Rule and interpret the result as an estimate of area.
Approximate a definite integral using the Trapezoidal Rule.	I can approximate a definite integral using the Trapezoidal Rule and explain how it estimates area using trapezoids.
Approximate a definite integral using Simpson's Rule.	I can approximate a definite integral using Simpson's Rule and apply the correct formula for an even number of subintervals.
Analyze the approximate errors in the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule.	I can analyze and compare the errors of the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule using concavity and known error behavior.

I deviate from the textbook slightly to include the Midpoint Rule and a small discussion. The theorem numbers will still match the textbook.

Now that we have discussed the Fundamental Theorem of Calculus, you might have asked yourself 'why would I use a Riemann sum if I can just take the antiderivative?'

First, some elementary functions do not have antiderivatives that are elementary functions. For example, there is no elementary function that has any of the following functions as its derivative

$$\sqrt[3]{x}\sqrt{1-x}, \quad \sqrt{x}\cos(x), \quad \frac{\cos(x)}{x}, \quad \sqrt{1-x^3}, \quad \sin(x^2), \quad e^{-x^2}$$

If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, the Fundamental Theorem of Calculus cannot be applied, and you must resort to an approximation technique.

Second, almost every integral is done numerically, even when it can be done symbolically (analytically) in a closed form way. Symbolic calculators, like Wolfram Alpha, will perform the necessary antiderivative. However, in practice, most of the time we just want a number. Performing the actual antiderivative is a complex process, whereas evaluating a function and multiplying is something a *computer* is designed to do. *Numerical Quadrature* (or numerical integration) is a way of calculating the integral approximately, but in a way that is "close enough".

4.6.1 The Midpoint Rule

The Midpoint rule is almost identical to the Left and Right Riemann Sum, except now it will shift the interval to be in between the left and right points. Recall the definition of a Riemann Sum.

Definition of a Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

For each subinterval $[x_{i-1}, x_i]$, calculate the midpoint m_i using the formula

$$m_i = \frac{x_{i-1} + x_i}{2}$$

Then form a rectangle for each subinterval. The area of the ith rectangle is

$$A = f(m_i)\Delta x$$

This implies that the sum of the areas of the n rectangles is

$$\begin{aligned} A &= f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x \\ &= \Delta x[f(m_1) + f(m_2) + \cdots + f(m_n)] \end{aligned}$$

Since f is continuous, the limit of this Riemann Sum as $n \to \infty$ is precisely the definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \int_a^b f(x)dx$$

Theorem: The Midpoint Rule

Let f be continuous on $[a, b]$. The Midpoint Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.35: Use the Midpoint Rule to estimate $\displaystyle\int_0^1 x^2 dx$ using four subintervals.

Start by computing the width of each subinterval. $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$. Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}\frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right]$$

The midpoints of each of these subintervals are halfway in between each value. For example, $\dfrac{\frac{1}{4}-0}{2} = \dfrac{1}{8}$. Therefore, the midpoints of the subintervals are $m_i = \left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned}
M_4 &= \sum_{i=1}^{n} f(m_i)\Delta x \\
&= \sum_{i=1}^{n} f(m_i)\left(\frac{1}{4}\right) \\
&= f\left(\frac{1}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{3}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{5}{8}\right)\cdot\left(\frac{1}{4}\right) + f\left(\frac{7}{8}\right)\cdot\left(\frac{1}{4}\right) \\
&= \left(\frac{1}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{9}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{25}{64}\right)\cdot\left(\frac{1}{4}\right) + \left(\frac{49}{64}\right)\cdot\left(\frac{1}{4}\right) \\
&= \left(\frac{1}{4}\right)\left[\left(\frac{1}{64}\right) + \left(\frac{9}{64}\right) + \left(\frac{25}{64}\right) + \left(\frac{49}{64}\right)\right] \\
&= \left(\frac{1}{4}\right)\left(\frac{21}{16}\right) \\
&= \frac{21}{64}
\end{aligned}$$

As you can see from the graph on the right, the function goes through the middle of each rectangle. So for a function that is increasing and concave up, the rectangles will have a slight overestimate on the left and a slight underestimate on the right.
You can see it for yourself on desmos.
Of course, by increasing the number of rectangles, you get more and more accurate results.

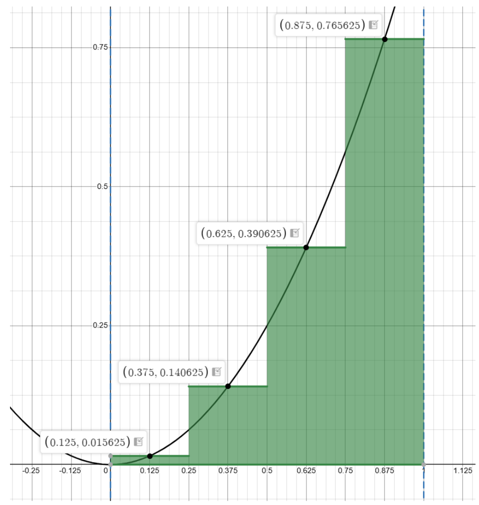


Figure 4.11

4.6.2 The Trapezoidal Rule

Another way to approximate a definite integral is to use n trapezoids, as shown in the figure below.

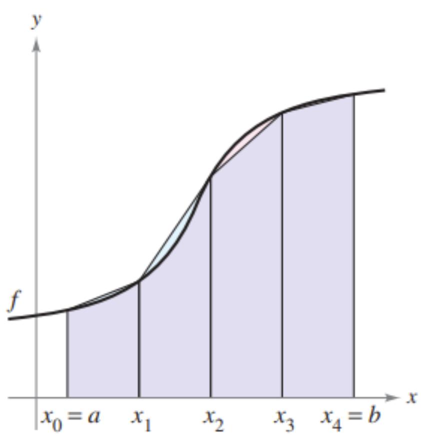


Figure 4.12 The area of a region can be approximated using four trapezoids.

In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Then form a trapezoid for each subinterval. The area of the ith trapezoid is

$$A = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right)$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned} A &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0)+f(x_1)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0)+f(x_1)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+f(x_n)] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)] \end{aligned}$$

Letting $\Delta x = (b-a)/n$, you can take the limit as $n \to \infty$ to obtain

$$\lim_{n\to\infty}\left(\frac{b-a}{2n}\right)[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

$$\begin{aligned} &= \lim_{n\to\infty}\left[\frac{[f(a)-f(b)]\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\Delta x\right] \\ &= \lim_{n\to\infty}\frac{[f(a)-f(b)]\Delta x}{2} + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= 0 + \int_a^b f(x)dx \end{aligned}$$

Theorem 4.16: The Trapezoidal Rule

Let f be continuous on $[a,b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{2n}\right)[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.36: Use the Trapezoidal Rule to estimate $\displaystyle\int_0^1 x^2\,dx$ using four subintervals.

Start by computing the width of each subinterval.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right].$$

The Trapezoidal Rule uses the endpoints of each subinterval. Thus, the partition points are

$$x_i = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

The trapezoidal rule with n subintervals is given by

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Applying this formula with $n = 4$, we have

$$\begin{aligned} T_4 &= \frac{1}{2}\left(\frac{1}{4}\right)\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \\ &= \frac{1}{8}\left[0 + 2\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{9}{16}\right) + 1\right] \\ &= \frac{1}{8}\left[\frac{2}{16} + \frac{2}{4} + \frac{18}{16} + 1\right] \\ &= \frac{1}{8}\left(\frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 1\right) \\ &= \frac{1}{8}\left(\frac{22}{8}\right) \\ &= \frac{11}{32}. \end{aligned}$$

As you can see from the graph on the right, we can form little trapezoids to approximate the function. In this case, since the function is increasing concave up, every estimate will be a slight overestimate.
You can see it for yourself on desmos.
Of course, by increasing the number of rectangles, you get more and more accurate results.

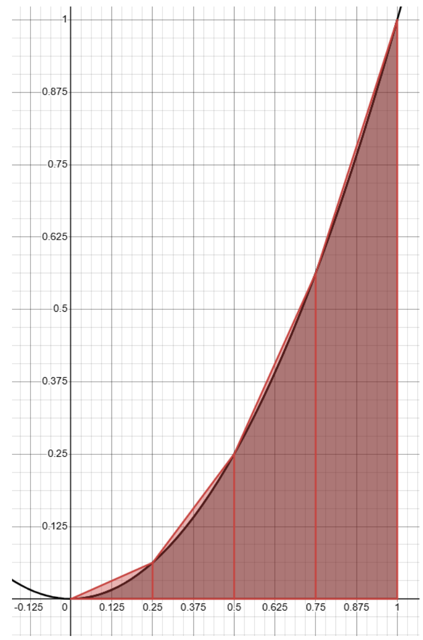


Figure 4.13

4.6.3 Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate $f$by a *first*-degree polynomial (that is, a line). In Simpson's Rule, named after the English mathematician Thomas Simpson (1710–1761), you take this procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, first we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

Theorem 4.17: Integral of $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{b-a}{2}\right) + p(b)\right]$$

Proof:

$$\begin{aligned}\int_a^b p(x)\,dx &= \int_a^b (Ax^2 + Bx + C)\,dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_a^b \\ &= \frac{A(b^3 - a^3)}{3} + \frac{B(b^2 - a^2)}{2} + C(b - a) \\ &= \left(\frac{b-a}{6}\right)\big[2A(a^2 + ab + b^2) + 3B(b + a) + 6C\big].\end{aligned}$$

By expansion and collection of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2 + Ba + C)}_{p(a)} + \underbrace{4\left[A\left(\frac{a+b}{2}\right)^2 + B\left(\frac{a+b}{2}\right) + C\right]}_{4p\left(\frac{a+b}{2}\right)} + \underbrace{(Ab^2 + Bb + C)}_{p(b)},$$

Therefore,

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

□

To develop Simpson's Rule for approximating a definite integral, you again partition the subinterval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$. This time, however, n is required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0, x_2]}\!\!\underbrace{\; < x_3 < x_4}_{[x_2, x_4]} < \cdots < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2}, x_n]} = b.$$

On each (double) subinterval $[x_{i-2}, x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2. For example, on the subinterval $[x_0, x_2]$, you can choose the polynomial of least degree passing through the points $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2). Now, using p as an approximation of f on this subinterval, you have, by Theorem 4.17,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx &= \left(\frac{x_2 - x_0}{6}\right)\left[p(x_0) + 4p\left(\frac{x_2 - x_0}{2}\right) + p(x_2)\right] \\ &= \frac{2[(b-a)/n]}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\ &= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)]\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

Theorem 4.18: Simpson's Rule (n is even)

Let f be continuous on $[a, b]$. Simpson's Rule for approximating $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]$$

Moreover, as $n \to \infty$ the right hand side approaches $\int_a^b f(x)dx$.

Example 4.37: Use Simpson's Rule to estimate $\displaystyle\int_0^1 x^2\,dx$ using four subintervals.
Start by computing the width of each subinterval.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right].$$

Simpson's Rule requires an even number of subintervals, which is satisfied here. The partition points are

$$x_i = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

The Simpson's Rule formula with n subintervals is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Applying this formula with $n = 4$, we obtain

$$\begin{aligned}
S_4 &= \frac{1}{3}\left(\frac{1}{4}\right)\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] \\
&= \frac{1}{12}\left[0 + 4\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{9}{16}\right) + 1\right] \\
&= \frac{1}{12}\left[\frac{4}{16} + \frac{2}{4} + \frac{36}{16} + 1\right] \\
&= \frac{1}{12}\left(\frac{1}{4} + \frac{1}{2} + \frac{9}{4} + 1\right) \\
&= \frac{1}{12}(4) \\
&= \frac{1}{3}.
\end{aligned}$$

Now, however, it should be obvious that this will be exact. This is because we are approximating a degree 2 polynomial with a degree 2 polynomial. However, I wanted to provide the same function in each example so the work could be consistently displayed.
You can see it for yourself on desmos.

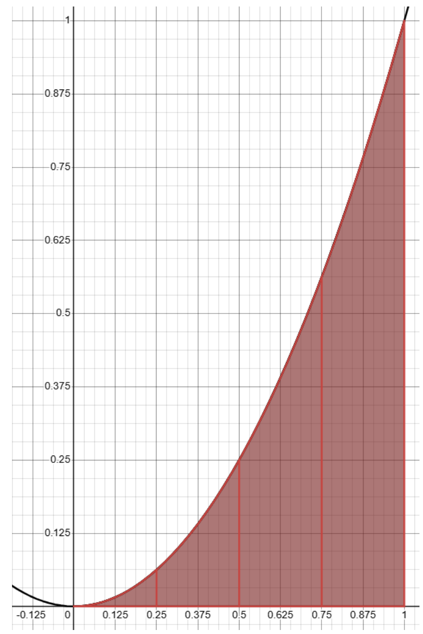


Figure 4.14

Here is a different example where the approximation is not exact.

Example 4.38: Use Simpson's Rule to estimate $\int_0^{\pi} \sin(x)\, dx$ using four subintervals.
Start by computing the width of each subinterval.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

Therefore, the subintervals are

$$\left[0, \frac{\pi}{4}\right], \quad \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \quad \left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \quad \left[\frac{3\pi}{4}, \pi\right].$$

The partition points are

$$x_i = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}.$$

Simpson's Rule formula with n subintervals is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right].$$

Applying this formula with $n = 4$, we have

$$\begin{aligned} S_4 &= \frac{\pi/4}{3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \\ &= \frac{\pi}{12}\left[0 + 4\cdot\frac{\sqrt{2}}{2} + 2\cdot 1 + 4\cdot\frac{\sqrt{2}}{2} + 0\right] \\ &= \frac{\pi}{12}\left[4\cdot\frac{\sqrt{2}}{2} + 2 + 4\cdot\frac{\sqrt{2}}{2}\right] \\ &= \frac{\pi}{12}\left[4\cdot\frac{\sqrt{2}}{2} + 4\cdot\frac{\sqrt{2}}{2} + 2\right] \\ &= \frac{\pi}{12}\left[4\sqrt{2} + 2\right] \\ &= \frac{\pi}{12}(2 + 4\sqrt{2}) \\ &= \frac{\pi}{6}(1 + 2\sqrt{2}) \\ &\approx 2.005 \end{aligned}$$

The approximation shows a slight underestimation near the top, but an overestimation on the sides.
You can see it for yourself on desmos.

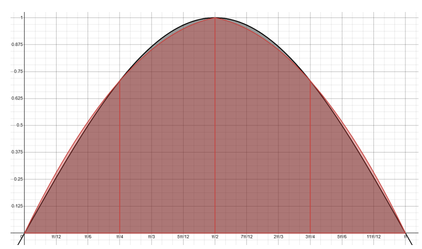


Figure 4.15

4.6.4 Error Analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. The following theorem, which is listed without proof, gives the formulas for estimating the errors involved in use of the Midpoint Rule, Simpson's Rule, and the Trapezoidal Rule.

Theorem 4.19: Errors in the Midpoint Rule, Simpson's Rule, and the Trapezoidal Rule.

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by the Midpoint Rule is

$$E \leq \frac{(b-a)^3}{24n^2}[\max|f''(x)|], \quad a \leq x \leq b$$

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by the Trapezoidal Rule is

$$E \leq \frac{(b-a)^3}{12n^2}[\max|f''(x)|], \quad a \leq x \leq b$$

If f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)dx$ by Simpson's Rule is

$$E \leq \frac{(b-a)^5}{180n^4}[\max|f^{(4)}(x)|], \quad a \leq x \leq b$$

Theorem 4.19 states that the errors generated by the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$ and $f^{(4)}(x)$ in the interval $[a, b]$ Furthermore, these errors can be made arbitrarily small by increasing n, provided that the functions are continuous and therefore bounded in $[a, b]$.

Practice Exercises

Approximating Definite Integrals

66. Use the Midpoint Rule with $n = 4$ subintervals to approximate the integral $\displaystyle\int_0^4 \sqrt{x^2+1}\,dx$. Round your answer to three decimal places.

67. Use the Trapezoidal Rule with $n = 4$ to approximate the value of $\displaystyle\int_1^3 \frac{1}{x}\,dx$. Show the expansion of the sum.

68. The function f is continuous, positive, and concave up on the interval $[a, b]$.

1. Does the Trapezoidal Rule approximation for $\int_a^b f(x)\,dx$ yield an overestimate or an underestimate? Explain geometrically.

2. Does the Midpoint Rule approximation for $\int_a^b f(x)\,dx$ yield an overestimate or an underestimate?

69. Selected values of a continuous function f are given in the table below.

x	0	2	4	6	8
$f(x)$	12	7	5	8	10

1. Use the Trapezoidal Rule with 4 subintervals of equal length to approximate $\int_0^8 f(x)\,dx$.

2. Use the Midpoint Rule with 2 subintervals of equal length to approximate $\int_0^8 f(x)\,dx$.

70. A radar gun measures the velocity $v(t)$ (in ft/sec) of a car at 5-second intervals.

t (sec)	0	5	10	15	20
$v(t)$	0	30	50	70	80

Estimate the total distance traveled by the car from $t = 0$ to $t = 20$ seconds using the Trapezoidal Rule with 4 subintervals.

71. Consider the integral $I = \displaystyle\int_0^2 e^{-x^2}\,dx$. If T_n represents the Trapezoidal Rule approximation and M_n represents the Midpoint Rule approximation with n subintervals, arrange T_4, M_4, and I in increasing order. (Note: Consider the concavity of $f(x) = e^{-x^2}$).

72. (Simpson's Rule) Use Simpson's Rule with $n = 4$ subintervals to approximate $\int_0^4 x^3\,dx$. Compare your result with the exact value of the integral.

73. (Simpson's Rule) The table below shows values of the function $g(x)$.

x	1	2	3	4	5	6	7
$g(x)$	2	5	8	6	4	7	3

Approximate $\int_1^7 g(x)\,dx$ using Simpson's Rule with $n = 6$ subintervals.