# Chapter 5 Transcendental Functions and Their Inverses

### Student Notes


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## 5.1 The Natural Logarithmic Function: Differentiation

**Lesson Objectives & Success Criteria**

| **Key Topics & Formulas** | **Success Criteria** |
|---|---|
| Develop and use properties of the natural logarithmic function | I can use the properties of natural logarithms to rewrite, simplify, and solve expressions. |
| Understand the definition of the number $e$ | I can explain what the number $e$ represents and how it is defined. |
| Find derivatives of functions involving the natural logarithmic function | I can find derivatives of functions that involve the natural logarithm. |

### 5.1.1 The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq 1$$

has an important disclaimer - it doesn't apply when $n = 1$.

Earlier in the course, when we covered **Chapter 2**, I specifically gave you the rules that covered this. Namely,

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

therefore

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln(x) + C$$

In this chapter, we will discuss the **natural logarithm function**. Being neither algebraic nor trigonometric, it is in its own class called *logarithmic functions*.

**Definition of the Natural Logarithmic Function**

The **natural logarithmic function** is defined by:

$$\ln(x) = \int_1^x \frac{1}{t} dt, \quad x > 0$$

The domain is the set of all positive real numbers $(0, \infty)$.

**Visualizing the Function:** Based on the definition as an accumulation function, we can determine:

- $\ln(x)$ is __________ for $x > 1$.
- $\ln(x)$ is __________ for $0 < x < 1$.
- $\ln(1) = 0$.

**Theorem 5.1**: Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties:

1. **Domain:** $(0, \infty)$ **Range:** $(-\infty, \infty)$
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

We can use properties of integrals to derive algebraic properties of logarithms.

**Theorem 5.2**: Logarithmic Properties

If $a$ and $b$ are positive numbers and $n$ is rational, then:

1. $\ln(1) = 0$
2. $\ln(ab) = \ln(a) + \ln(b)$ (Product Property)
3. $\ln(a^n) = n\ln(a)$ (Power Property)
4. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ (Quotient Property)

**Example 5.1**: Use properties of logarithms to rewrite and simplify the expressions:

a) $\ln\left(\dfrac{10}{9}\right)$

b) $\ln\left(\sqrt{3x+2}\right)$

c) $\ln\left(\dfrac{6x}{5}\right)$

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original. For instance, the domain of $f(x) = \ln(x^2)$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln(x)$ is all positive real numbers.

## Practice Exercises

**Use properties to expand the expression**

**1.** $\ln\left(\frac{5}{6}\right)$

**2.** $\ln\left(\sqrt{2^3}\right)$

**3.** $\ln\left(\sqrt[3]{b^2+2}\right)$

**4.** $\ln\left(z(z-1)^2\right)$

### 5.1.2 The Number $e$

In algebra, you learned that logarithms require a **base**.

- Common Log: Base 10 ($\log_{10} x$)
- Natural Log: Base $e$ ($\ln x = \log_e x$)

**Definition of $e$**

The letter $e$ denotes the positive real number such that:

$$\ln(e) = \int_1^e \frac{1}{t} dt = 1$$

Approximation: $e \approx 2.71828...$

Once you know that $\ln(e) = 1$, you can use logarithmic proeprties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned} \ln(e^n) &= n \ln(e) \\ &= n(1) \\ &= n \end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of $n$, as shown below.

| $x$ | $\frac{1}{e^3} \approx 0.050$ | $\frac{1}{e^2} \approx 0.135$ | $\frac{1}{e} \approx 0.368$ | $e^0 = 1$ | $e \approx 2.718$ | $e^2 \approx 7.389$ |
|---|---|---|---|---|---|---|
| $\ln(x)$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 |

The logarithms shown in the table above are convenient because the x-values are integer powers of $e$. Most logarithmic expressions are, however, best evaluated with a calculator.

**Example 5.2**:

1. $\ln(2) \approx .693$
2. $\ln(32) \approx 3.466$
3. $\ln(0.1) \approx -2.303$

### 5.1.3 The Derivative of the Natural Logarithmic Function

**Theorem 5.3**: Derivative of the Natural Logarithmic Function

Let $u$ be a differentiable function of $x$.

1. $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}, \quad x > 0$
2. $\dfrac{d}{dx}[\ln(u)] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

We have already discussed this derivative and used it before. However, now we can see the first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

**Example 5.3**: Let $u = x^2 + 1$

It is often easier to apply the rules of logarithms first, then differentiate.

**Example 5.4**: Differentiate $f(x) = \ln(\sqrt{x+1})$.
We can use the properties of logarithms to rewrite:

and now differentiate:

On occasion, it is convenient to use logarithms as aids in differentiating nonlogarithmic functions. This procedure is called **logarithmic differentiation**.

**Example 5.5**: Find the derivative of $y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$
Note that $y > 0$ for all $x \neq 2$. So, $\ln(y)$ is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for $y'$.

Write original equation.

Take natural log of each side.

Logarithmic properties

Differentiate.

Solve for $y'$.

Substitute for $y$ and get common denominator.

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{\frac{3}{2}}}$$ Simplify.

**Derivatives Involving Absolute Value**

Since $\ln(x)$ is only defined for $x > 0$, we often work with $\ln|u|$.

The following theorem states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value sign were not present.

**Theorem 5.4**: Derivative involving Absolute Value

If $u$ is a differentiable function of $x$ such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

**Example 5.6**: Find the derivative of $f(x) = \ln|\cos(x)|$.

## Practice Exercises

**Find the derivative of each of the functions.**

**5.** $g(x) = \ln(x^2)$

**6.** $y = x\ln(x)$

**7.** $y = \ln|\sin(x)|$

**8.** $f(x) = \ln\left(\frac{2x}{x+2}\right)$

**9.** $y = \ln\left(\frac{\sqrt{4+x^2}}{x}\right)$ *Hint: Use log properties*

## 5.2 The Natural Logarithmic Function: Integration

**Lesson Objectives & Success Criteria**

| Key Topics & Formulas | Success Criteria |
|---|---|
| Use the Log Rule for Integration to integrate a rational function. | I can recognize when a rational function fits the Log Rule and write its integral. |
| Integrate trigonometric functions. | I can choose the correct antiderivative for basic trigonometric functions |

### 5.2.1 Log Rule for Integration

Recall that the Power Rule $\int x^n dx$ does not apply when $n = -1$. For this case, we use the Log Rule.

**Theorem 5.5**: Log Rule for Integration

Let $u$ be a differentiable function of $x$.

1. $\int \frac{1}{x} dx = \ln|x| + C$
2. $\int \frac{1}{u} du = \ln|u| + C$

**Alternate Form:** Since $du = u' dx$, we can often recognize the pattern:

$$\int \frac{u'}{u} dx = \ln|u| + C$$

*Strategy: If the numerator is the derivative of the denominator, the integral is the natural log of the denominator.*

**Example 5.7**: Integrate $\int \frac{2}{x} dx$

Because $x^2$ cannot be negative, the absolute value is unnecessary in the final form of the antiderivative.

**Example 5.8**: Find $\int \frac{1}{4x-1} dx$.

The next example uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

**Example 5.9**: Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2+1}$$

the $x$-axis, and the line $x = 3$.

**Guidelines for Integration Strategies**

1. **Check Basic Formulas:** Is it a simple Power Rule or Trig Rule?
2. **Check Log Rule ($u'/u$):** Is the numerator the derivative (or constant multiple) of the denominator?
3. **Integration by Substitution:** Find a $u$ such that $du$ is present in the integrand.
4. **Algebraic Manipulation:**
   - Trig Identities
   - Multiply/Divide by a "clever form of 1"

**Example 5.10**: Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x\ln(x)}$.

### 5.2.2 Integrals of Trigonometric Functions

Previously, we only knew integrals for $\sin x$ and $\cos x$ (and things like $\sec^2 x$). Now, we can find the integrals for $\tan x$, $\cot x$, $\sec x$, and $\csc x$ by rewriting them as rational functions.

**Example 5.11**: Find $\int \tan(x)dx$.

The previous example uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

**Example 5.12**: Find $\int \sec(x)dx$.

With the results of the last two examples, you now have integration formulas for all six trigonometric functions.

**Integrals of the Six Basic Trigonometric Functions**

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \cos(u)\,du = \sin(u) + C$$

$$\int \tan(u)\,du = -\ln|\cos(u)| + C \qquad \int \cot(u)\,du = \ln|\sin(u)| + C$$

$$\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C \qquad \int \csc(u)\,du = -\ln|\csc(u) + \cot(u)| + C$$

**Example 5.13**: Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}dx$

## Practice Exercises

Evaluate the following definite and indefinite integrals

**10.** $\int \cot(4x)\,dx$

**11.** $\displaystyle\int \frac{\tan(\ln(x))}{x}\,dx$

**12.** $\displaystyle\int e^x \sec(e^x)\,dx$

**13.** $\displaystyle\int x\csc(x^2)\,dx$

**14.** $\displaystyle\int_0^{\frac{\pi}{3}} \tan(x)\,dx$

**15.** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec(2t)\,dt$

## 5.3 Inverse Functions

**Lesson Objectives & Success Criteria**

| **Key Topics & Formulas** | **Success Criteria** |
|---|---|
| Verify that one function is the inverse function of another function. | I can show that two functions are inverses by composing them and obtaining the identity function. |
| Determine whether a function has an inverse function. | I can determine whether a function has an inverse by checking if it is one-to-one. |
| Find the derivative of an inverse function. | I can find the derivative of an inverse function using the inverse function theorem. |

### 5.3.1 Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f : \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of $f$. This function is denoted by $f^{-1}$. It is a function from $B$ to $A$, and can be written as

$$f^{-1} : \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that of $f$ is equal to the range of $f^{-1}$, and vice versa. The functions $f$ and $f^{-1}$ have the effect of "undoing" each other. That is, when you form the composition of $f$ with $f^{-1}$ or the composition of $f^{-1}$ with $f$. you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

**Definition of Inverse Function**

A function $g$ is the **inverse function** of the function $f$ if:

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f$$

The function $g$ is denoted by $f^{-1}$ (read "$f$ inverse").

**Example 5.14**: Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Because the domains and ranges of both $f$ and $g$ consist of all real numbers, you can conclude that both composite functions exist for all $x$. The composition of $f$ with $g$ is given by

The composition of $g$ with $f$ is given by

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that $f$ and $g$ are inverse functions of each other.

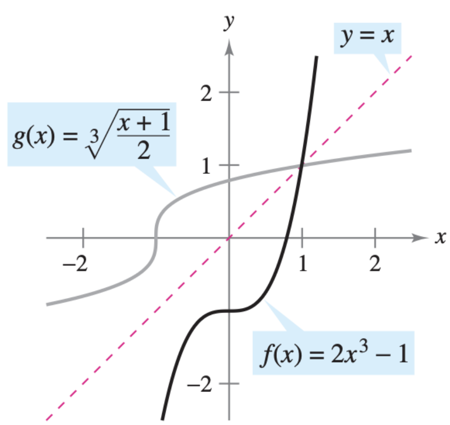



Figure 5.1 $f$ and $g$ are inverse functions of each other

Notice from the figure that the graphs are **reflections** of one another over the line $y = x$.

**Theorem 5.6**: Reflective Property of Inverse Functions

The graph of $f$ contains the point $(a, b)$ if and only if the graph of $f^{-1}$ contains the point $(b, a)$.

### 5.3.2 Existence of an Inverse Function

Not every function has an inverse. For $f^{-1}$ to be a function, the original function $f$ must pass the **Horizontal Line Test**.

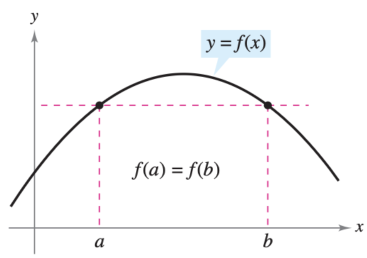



Figure 5.2 Does not pass the Horizontal Line Test

**Theorem 5.7**: The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If $f$ is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

**Example 5.15**: Which of the functions has an inverse function?

(a) $f(x) = x^3 + x - 1$

(b) $f(x) = x^3 - x + 1$

By graphing, you can see that (a) increases over its entire domain, whereas (b) increases, decreases, then increases again. Therefore, (b) does not pass the horizontal line test. In other words, it is not one-to-one. Therefore, by Theorem 5.7, $f(x) = x^3 - x + 1$ does not have an inverse.

### Finding the Inverse Equation

**Guidelines for Finding an Inverse Function**

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Interchange (swap) $x$ and $y$.
3. Solve the equation for $y$. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of $f^{-1}$ to be the range of $f$.
5. Verify that $f(f^{-1}(x))$ and $f^{-1}(f(x)) = x$.

**Example 5.16**: Find the inverse function of $f(x) = \sqrt{2x - 3}$.

Theorem 5.7 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

**Example 5.17**: Show that the sine function, $f(x) = \sin(x)$ is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, for which $f$ is strictly monotonic.
It is clear that $f$ is not one-to-one, because many different $x$-values yield the same $y$-values. For instance,

$$\sin(0) = 0 = \sin(\pi)$$

Moreover, $f$ is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos(x)$$

is positive there.
Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that $f$ is increasing on the closed interval $(-\pi/2, \pi/2)$ *and* that in any larger interval the function is not strictly monotonic.

## Practice Exercises

**16.** Use the derivative to determine if the function $f(x) = 2x^3 + 5x - 3$ has an inverse function on the entire real line. Explain your reasoning.

**17.** Determine whether the function $f(x) = x^4 - 2x^2$ has an inverse function on the interval $(-\infty, \infty)$. If not, find a restricted domain $x \geq c$ where the function would have an inverse.

**18.** Find the inverse function $f^{-1}(x)$ for the function $f(x) = \sqrt[3]{3x-5}$.

**19.** Find the inverse function of the rational function $f(x) = \frac{x+2}{3x-1}$.

**20.** The function $f(x) = \sqrt{x-4}$ is defined on the domain $[4, \infty)$. Find the equation for $f^{-1}(x)$ and state the domain of $f^{-1}$.

**21.** Let $f(x) = (x-2)^2$ for $x \leq 2$. Find an expression for $f^{-1}(x)$.

### 5.3.3 Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions as shown in Figure 5.12.

Since the graphs are reflections over $y = x$, the slopes are **reciprocals**.

**Theorem 5.8**: Continuity and Differentiability of Inverse Functions

Let $f$ be a function whose domain is an interval $I$. If $f$ has an inverse function, then the following statements are true.

1. If $f$ is continuous on its domain, then $f^{-1}$ is continuous on its domain.
2. If $f$ is increasing on its domain, then $f^{-1}$ is increasing on its domain.
3. If $f$ is decreasing on its domain, then $f^{-1}$ is decreasing on its domain.
4. If $f$ is differentiable at $c$ and $f'(c) \neq 0$, then $f^{-1}$ is differentiable at $f(c)$.

This is the most critical Calculus concept for this section. We can find the slope of the inverse function *without* actually finding the inverse equation.

**Theorem 5.9**: The Derivative of an Inverse Function

Let $f$ be a function that is differentiable on an interval $I$. If $f$ has an inverse function $g$, then $g$ is differentiable at any $x$ for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

**Example 5.18**: Let $f(x) = \frac{1}{4}x^3 + x - 1$.

1. What is the value of $f^{-1}(3)$?

2. What is the value of $(f^{-1})'(3)$?

In the previous example, note that the point $(2,3)$ the slope of the graph of $f$ is 4 and at the point $(3,2)$ the slope of the graph of $f^{-1}$ is $\frac{1}{4}$. This reciprocal relationship (which follows from Theorem 5.9) can be written as shown below.
If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Theorem 5.9 says that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}$$

So,

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

## Practice Exercises

**22.** Given $f(x) = x^3 + 2x - 1$. Notice that $f(1) = 2$. Find the value of $(f^{-1})'(2)$.

**23.** Let $f$ be the function defined by $f(x) = \sqrt{x-4}$. Find the value of the derivative of $f^{-1}$ at $x = 2$.

**24.** Let $g(x) = f^{-1}(x)$. The table below gives values of the differentiable one-to-one function $f$ and its derivative $f'$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 4 | 6 | 2 | 5 |
| $f'(x)$ | 5 | $\frac{1}{2}$ | -3 | 4 |

Using the table, calculate $g'(6)$.

**25.** The function $h$ is differentiable and one-to-one. If the point $(3, 8)$ lies on the graph of $h$ and $h'(3) = \frac{4}{5}$, find the slope of the tangent line to the graph of $h^{-1}$ at $x = 8$.

**26.** Let $f(x) = \sin(x)$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find $(f^{-1})'(\frac{\sqrt{3}}{2})$.

**27.** Let $f(x) = e^x + x$. Find the value of $(f^{-1})'(1)$.

## 5.4 Exponential Functions: Differentiation and Integration

**Lesson Objectives & Success Criteria**

| Key Topics & Formulas | Success Criteria |
|---|---|
| Develop properties of the natural exponential function. | I can explain the key properties of the natural exponential function and use them to rewrite expressions. |
| Differentiate natural exponential functions. | I can find the derivative of functions involving $e^x$ and apply the result in context. |
| Integrate natural exponential functions. | I can evaluate integrals involving $e^x$ and use them to solve problems. |

### 5.4.1 The Natural Exponential Function

Since $f(x) = \ln(x)$ is increasing and one-to-one, it must have an inverse function.

**Definition of the Natural Exponential Function**

The inverse function of the natural logarithmic function $f(x) = \ln(x)$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln(y)$$

**Inverse Relationships:**

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x$$

**Example 5.19**: Solve $7 = e^{x+1}$.

**Example 5.20**: Solve $\ln(2x-3)=5$.

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the following theorem.

**Theorem 5.10**: Operations with Exponential Functions

Let $a$ and $b$ be any real numbers.

1. $e^a \cdot e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$

In Section 5.3, you learned that an inverse function $f^{-1}$ shares many properties with $f$. So, the natural exponential function inherits the following properties from the natural logarithmic function. See the figure below:

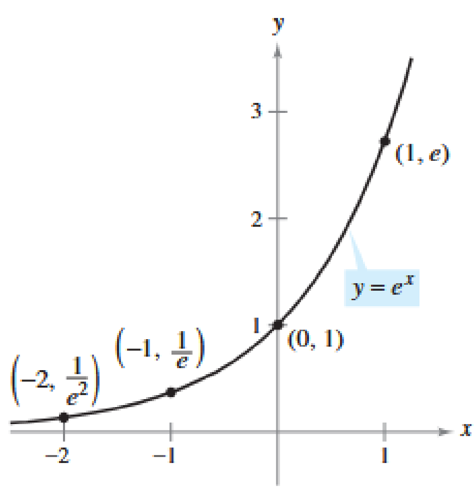



Figure 5.3 The natural exponential function is increasing, and its graph is concave upward.

**Properties of the Natural Exponential Function**

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

### 5.4.2 Derivatives of Exponential Functions

The natural exponential function is unique because *it is its own derivative.* It is the solution to the differential equation $y' = y$.

**Theorem 5.11**: Derivative of the Natural Exponential Function

Let $u$ be a differentiable function of $x$.

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[e^u] = e^u \cdot u' = e^u \frac{du}{dx}$ **(Chain Rule)**

Since we have already been using this derivative, we will not be going through a bunch of examples. However, below is one example which I think will be fun for the people taking statistics.

**Example 5.21**: Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.
To locate possible points of inflection, find the $x$-values for which the second derivative is 0.

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\
f'(x) &= \frac{1}{\sqrt{2\pi}} (-x) e^{-x^2/2} \\
f''(x) &= \frac{1}{\sqrt{2\pi}} \left[ (-x)(-x) e^{-x^2/2} + (-1) e^{-x^2/2} \right] && \text{Product Rule} \\
&= \frac{1}{\sqrt{2\pi}} (e^{-x^2/2})(x^2 - 1)
\end{aligned}$$

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 4 to conclude that these values yield two points of inflection as shown in the figure.

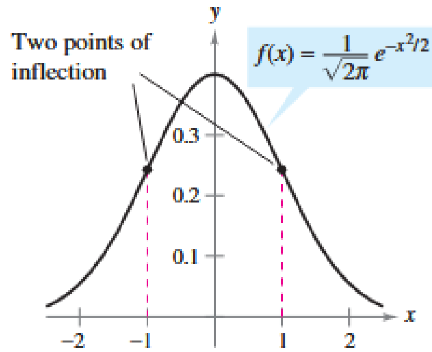



Figure 5.4 The bell-shaped curve given by a standard normal probability density function

As another note, the probability density function for a normal distribution with mean $\mu = 0$ is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

where $\sigma$ is the standard deviation. This "bell-shaped curve" has points of inflection at $x = \pm\sigma$. The region below the curve has a total area equal to 1, representing the entire sample space. Computing the integral on the interval $(-\infty, z)$ yields the **cumulative probability** (the value found in the body of standard lookup tables) associated with the given $z$-score:

$$P(X < z) = \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\, dx$$

## Practice Exercises

Find the derivative of the following functions

**28.** $y = e^{-x^2}$

**29.** $f(x) = x^3 e^{4x}$

**30.** $g(t) = \frac{e^t}{e^t+2}$

**31.** Find the equation of the tangent line to the graph of $y = (x-1)e^x$ at the point where $x = 1$.

**32.** Find $\frac{dy}{dx}$ implicitly for the equation $e^{xy} + x = y^2$.

**33.** Find the $x$-coordinate of the point where the tangent line to the graph of $f(x) = 2xe^{-x}$ is horizontal.

### 5.4.3 Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

**Theorem 5.12**: Integration Rules for Exponential Functions

Let $u$ be a differentiable function of $x$.

1. $\displaystyle\int e^x dx = e^x + C$
2. $\displaystyle\int e^u du = e^u + C$

**Example 5.22**: Find $\displaystyle\int e^{3x+1} dx$.

**Example 5.23**: Find $\displaystyle\int 5xe^{-x^2} dx$.

**Example 5.24**: Evaluate the definite integral $\int_{-1}^{0} e^x \cos(e^x)dx$.

## Practice Exercises

**34.** Find the indefinite integral $\int e^{4-2x}\,dx$.

**35.** Find $\int x^2 e^{x^3+1}\,dx$.

**36.** Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

**37.** Find $\int (e^x + e^{-x})^2\,dx$.

**38.** Evaluate the definite integral $\int_0^{\ln(2)} e^{3x}\,dx$.

**39.** Evaluate $\int e^x \sin(e^x)\,dx$.

## 5.5 Bases Other than $e$ and Applications

**Lesson Objectives & Success Criteria**

| Key Topics & Formulas | Success Criteria |
|---|---|
| Define exponential functions that have bases other than $e$. | I can define and rewrite exponential functions with any positive base using properties of exponents and logarithms. |
| Differentiate and integrate exponential functions that have bases other than $e$. | I can differentiate and integrate exponential functions with bases other than $e$ by rewriting them in terms of $e$. |
| Use exponential functions to model compound interest and exponential growth. | I can use exponential functions to model compound interest and real-world exponential growth situations. |

### 5.5.1 Bases Other than $e$

The **base** of the natural exponential function is $e$. This "natural" base can be used to assign a meaning to a general base $a$.

**Definition of Exponential Function to Base $a$**

If $a$ is a positive real number ($a \neq 1$) and $x$ is any real number, then the **exponential function to the base** $a$ is denoted by $a^x$ and is defined by

$$a^x = e^{\ln(a)x}$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

**Example 5.25**: The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

## Definition of Logarithmic Function to Base $a$

If $a$ is a positive real number ($a \neq 1$) and $x$ is any positive real number, then the **logarithmic function to the base** $a$ is denoted by $\log_a(x)$ and is defined as

$$\log_a(x) = \frac{1}{\ln(a)} \ln(x)$$

Logarithmic functions to the base $a$ have properties similar to those of the natural logarithmic function given in Theorem 5.2.

1. $\log_a(1) = 0$
2. $\log_a(xy) = \log_a(x) + \log_a(y)$
3. $\log_a(x^n) = n \log_a(x)$
4. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

From the definitions of the exponential and logarithmic functions to the base $a$, it follows that $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverse functions of each other.

## Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a(y)$
2. $a^{\log_a(x)} = x$, for all $x > 0$
3. $\log_a(a^x) = x$, for all $x$

The logarithmic function to the base 10 is called the **common logarithmic function**. So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10}(y)$.

**Example 5.26**: Solve for $x$ in each equation.

1. $3^x = \dfrac{1}{81}$

2. $\log_2(x) = -4$

### 5.5.2 Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of $a^x$ and $\log_a(x)$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than $e$.

- When **differentiating** $a^x$, we **multiply** by $\ln(a)$.
- When **integrating** $a^x$, we **divide** by $\ln(a)$.

**Theorem 5.13**: Derivatives for Bases Other than $e$

Let $a$ be a positive real number ($a \neq 1$) and let $u$ be a differentiable function of $x$.

1. $\dfrac{d}{dx}[a^x] = (\ln(a))a^x$
2. $\dfrac{d}{dx}[a^u] = (\ln(a))a^u \dfrac{du}{dx}$
3. $\dfrac{d}{dx}[\log_a(x)] = \dfrac{1}{(\ln(a))x}$
4. $\dfrac{d}{dx}[\log_a(u)] = \dfrac{1}{(\ln(a))u}\dfrac{du}{dx}$

Occasionally, an integrand involves an exponential function to a base other than $e$. When this occurs, there are two options: (1) convert to base $e$ using the formula $a^x = e^{\ln(a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x dx = \left(\frac{1}{\ln(a)}\right) a^x + C$$

(which follows from Theorem 5.13).

**Example 5.27**: Find the derivative of each function.

1. $y = 2^x$
2. $y = 2^{3x}$
3. $y = \log_{10} \cos(x)$

**Theorem 5.14**: Integration Rule for Bases Other than $e$

$$\int a^x dx = \frac{1}{\ln(a)} a^x + C$$

**Example 5.28**: Find $\int 2^x dx$.

When the Power Rule, $\frac{d}{dx}[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent $n$ was required to be a ration number. Now the rule is extended to cover any real value of $n$.

**Theorem 5.15**: The Power Rule for Real Exponents

1. $\frac{d}{dx}[x^n] = nx^{n-1}$
2. $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

## Practice Exercises

Evaluate the following derivatives and integrals

**40.** Find the derivative of the function $f(x) = 4^{3x-2}$.

**41.** Find $g'(t)$ given $g(t) = \log_5(\sqrt{t^2+1})$.

**42.** Differentiate $y = x^\pi + \pi^x$.

**43.** Evaluate the indefinite integral $\int x6^{x^2}\,dx$.

**44.** Evaluate the definite integral $\int_{-1}^{2} 2^x\,dx$.

**45.** Find the equation of the tangent line to the graph of $y = \log_2(x)$ at the point $(8, 3)$.

### 5.5.3 Applications of Exponential Functions

Suppose $P$ dollars is deposited in an account at an annual interest rate $r$ (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year? The answer depends on the number of times $n$ the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n$$

As $n$ increases, the balance $A$ approaches a limit. To develop this limit, use the following theorem.

**Theorem 5.16**: A Limit Involving $e$

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Now, looking back at the formula for the balance $A$ in an account as the limit $n$ goes to infinity, we see

$$\begin{aligned}
A &= \lim_{n\to\infty}\left[P\left(1+\frac{r}{n}\right)^n\right] && \text{Take limit as } n\to\infty. \\
&= P\lim_{n\to\infty}\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^r && \text{Rewrite.} \\
&= P\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^r && \text{Let } x = n/r.\text{ Then } x\to\infty \text{ as } n\to\infty \\
&= Pe^r && \text{Apply Theorem 5.15}
\end{aligned}$$

This limit produces the balance after 1 year of **continuous compounding**. So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$\begin{aligned}
A &= 1000e^{0.08} \\
&\approx \$1083.29
\end{aligned}$$

**Example 5.29**: Calculate the balance (to the nearest cent) of a \$1,500 deposit at 6% interest after 1 year if compounded:

(a) Monthly ($n = 12$):

(b) Continuously:

Populations in real-world biology seldom grow exponentially indefinitely; they are constrained by factors such as food and space. This phenomenon is effectively represented by **Logistic Growth**.

**Example 5.30**: A bacterial culture is growing according to the *logistic growth function*

$$y = \frac{1.25}{1+0.25e^{-.4t}}, \quad t \geq 0$$

where $y$ is the weight of the culture in grams and $t$ is the time in hours. Find the weight of the culture after (a) 0 hours, (b) 1 hour, and (c) 10 hours. (d) What is the limit as $t$ approaches infinity?

## Practice Exercises

**46.** Evaluate the following limit: $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$.

**47.** A sum of \$5,000 is invested at an annual interest rate of 6%. Find the balance in the account after 10 years if the interest is compounded continuously.

**48.** The spread of a flu virus in a school is modeled by the function $P(t) = \frac{1200}{1+19e^{-0.6t}}$, where $P(t)$ is the number of infected students $t$ days after the virus is first identified.

1. How many students are initially infected?

2. What is the carrying capacity of the model (the maximum number of stu-

dents who will eventually be infected)?

**49.** A radioactive substance decays according to the formula $A(t) = A_0 e^{-0.05t}$, where $t$ is measured in years. If the initial amount $A_0$ is 100 grams, find the rate of decay $\frac{dA}{dt}$ when $t = 10$ years.

**50.** Find the equation of the tangent line to the graph of the logistic function $y = \frac{20}{1+3e^{-2x}}$ at $x = 0$.

**51.** The population of a city is given by $P(t) = 50,000e^{0.02t}$, where $t = 0$ represents the year 2000. Find the instantaneous rate of change of the population in the year 2010.

## 5.6 Inverse Trigonometric Functions: Differentiation

**Lesson Objectives & Success Criteria**

| **Key Topics & Formulas** | **Success Criteria** |
|---|---|
| Develop properties of the six inverse trigonometric functions. | I can describe the domains, ranges, and key properties of the inverse trigonometric functions. |
| Differentiate an inverse trigonometric function. | I can find the derivative of an inverse trigonometric function and apply it to solve problems. |
| Review the basic differentiation rules for elementary functions. | I can correctly apply the basic differentiation rules to elementary functions. |

### 5.6.1 Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains.*

In an example in Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$. On this interval you can define the inverse of the *restricted* sine function to be

$$y = \arcsin(x) \quad \text{if and only if} \quad \sin(y) = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq \arcsin(x) \leq \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

## Definitions of Inverse Trigonometric Functions

| Function | Domain | Range |
|---|---|---|
| $y = \arcsin(x) \iff \sin(y) = x$ | $-1 \le x \le 1$ | $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ |
| $y = \arccos(x) \iff \cos(y) = x$ | $-1 \le x \le 1$ | $0 \le y \le \pi$ |
| $y = \arctan(x) \iff \tan(y) = x$ | $-\infty < x < \infty$ | $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ |
| $y = \text{arccot}(x) \iff \cot(y) = x$ | $-\infty < x < \infty$ | $0 < y < \pi$ |
| $y = \text{arcsec}(x) \iff \sec(y) = x$ | $\lvert x\rvert \ge 1$ | $0 \le y \le \pi,\ y \ne \dfrac{\pi}{2}$ |
| $y = \text{arccsc}(x) \iff \csc(y) = x$ | $\lvert x\rvert \ge 1$ | $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2},\ y \ne 0$ |

Note that $\iff$ stands for "if and only if", which is a bidirectional statement in math. Meaning that the statement is true in both directions (each implies the other).

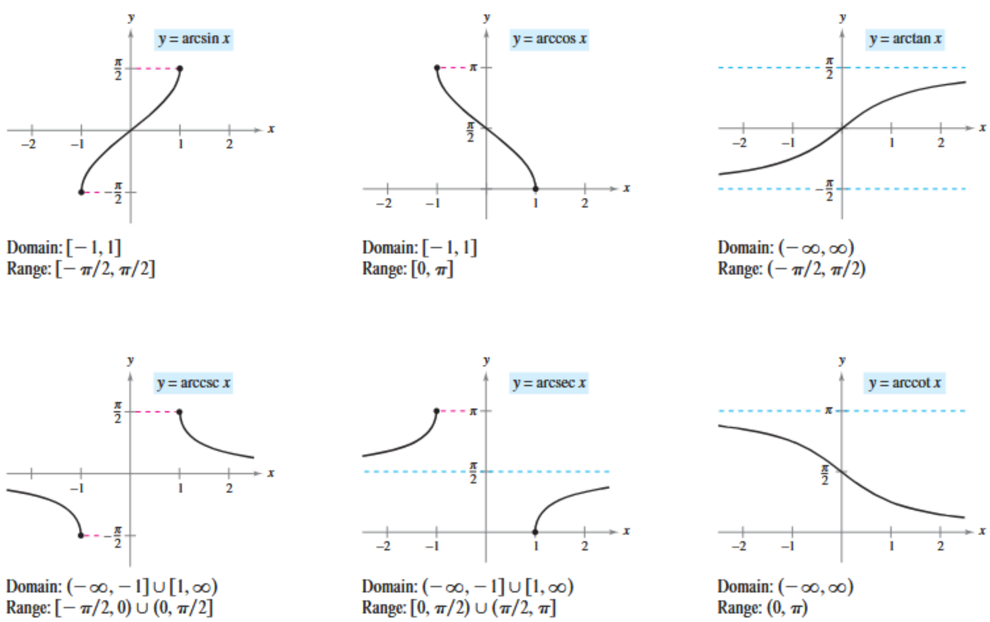
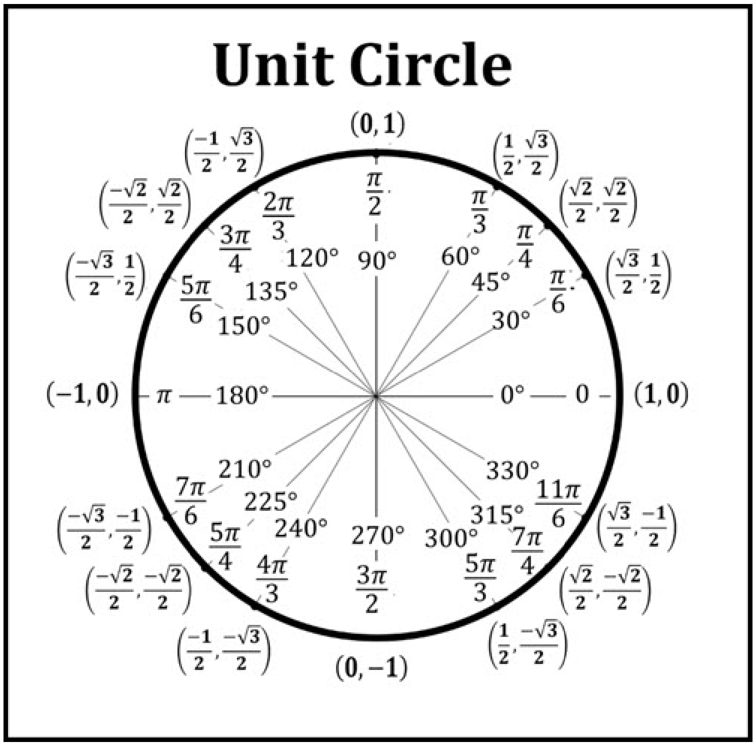



Figure 5.5 Graphs of the six inverse trigonometric functions.

Here, I will provide a unit circle because it will be helpful for our examples.




However, I would like to note that there is no unit circle provided on the AP Calculus test, nor is there for the final exam. **You will need to know the values of the unit circle.**

**Example 5.31**: Evaluate each function.

1. $\arcsin\left(-\frac{1}{2}\right)$

2. $\arccos(0)$

3. $\arctan\left(\sqrt{3}\right)$

4. arcsin(0.3) *This requires a calculator in **radian mode**.*

Inverse functions have the properties

$$f(f{-}1(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For $x$-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin(\pi)) = 0$, not $\pi$.

**Properties of Inverse Trigonometric Functions**

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$. then

$$\sin(\arcsin(x)) = x \quad \text{and} \quad \arcsin(\sin(y)) = y$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan(x)) = x \quad \text{and} \quad \arctan(\tan(y)) = y$$

If $|x| \geq 1$ and $0 \leq y \leq \pi/2$ or $\pi/2 < y \leq \pi$, then

$$\sec(\operatorname{arcsec}(x)) = x \quad \text{and} \quad \operatorname{arcsec}(\sec(y)) = y$$

Similar properties hold for the other inverse trigonometric functions.

**Example 5.32**: Given $y = \operatorname{arcsec}\left(\sqrt{5}/2\right)$, find $\tan(y)$.

### 5.6.2 Derivatives of Inverse Trigonometric Functions

Even though inverse trig functions are transcendental, their derivatives are **algebraic** (involving square roots and polynomials).

**Theorem 5.17**: Derivatives of Inverse Trigonometric Functions

Let $u$ be a differentiable function of $x$.

$$\frac{d}{dx}[\arcsin(u)] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos(u)] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot}(u)] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec}(u)] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc}(u)] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

*Note: The "co-" functions are just the negatives of their partners.*

To derive these formulas, you can use implicit differentiation. For instance, if $y = \arcsin(x)$, then $\sin(y) = x$. Differentiating implicitly yields $\cos(y)y' = 1$.

NOTE: There is no common agreement on the definition of $\text{arcsec}(x)$ (or $\text{arccsc}(x)$ for negative values of $x$. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\text{arcsec}(x) = \arccos\left(\frac{1}{x^2}\right)$$

For example, to evaluate $\text{arcsec}(-2)$ you can write

$$\text{arcsec}(-2) = \arccos(-0.5) \approx 2.09$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every $x$-value in its domain. This accounts for the absolute value sign in the formula for the derivative of $\text{arcsec}(x)$.

**Example 5.33**:

1. $\frac{d}{dx}[\arcsin(2x)]$

2. $\frac{d}{dx}[\arctan(3x)]$

**Example 5.34**: Differentiate $y = \arcsin(x) + x\sqrt{1-x^2}$.

## Practice Exercises

Find the derivative of the following functions

**52.** $y = \arcsin(3x^2)$

**53.** $f(x) = \arctan(\sqrt{x})$

**54.** $g(t) = \operatorname{arcsec}(5t)$

**55.** Find the equation of the tangent line to the graph of $y = \arctan(x)$ at $x = 1$.

**56.** $y = x \arccos(2x)$

**57.** Show that the derivative of $y = \arcsin(x) + \arccos(x)$ is 0.

### 5.6.3 Review of Basic Differentiation Rules

As mathematics has developed during the past few hundred years, a small number of elementary functions has proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An elementary function is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

| *Algebraic Functions* | *Transcendental Functions* |
|---|---|
| Polynomial functions | Logarithmic functions |
| Rational functions | Exponential functions |
| Functions involving radicals | Trigonometric functions |
| | Inverse trigonometric functions |

With the differentiation rules introduced so far in the text, you can differentiate any elementary function. For convenience, these differentiation rules are summarized below.

## Basic Differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = c\,u'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}\,u', \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)\,a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$
23. $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24. $\frac{d}{dx}[\text{arccsc}\, u] = -\frac{u'}{|u|\sqrt{u^2-1}}$

## 5.7 Inverse Trigonometric Functions: Integration

**Lesson Objectives & Success Criteria**

| **Key Topics & Formulas** | **Success Criteria** |
|---|---|
| Integrate functions whose antiderivatives involve inverse trigonometric functions. | I can recognize integrals that lead to inverse trigonometric functions and evaluate them correctly. |
| Use the method of completing the square to integrate a function. | I can complete the square to rewrite an integrand and use it to evaluate the integral. |
| Review the basic integration rules involving elementary functions. | I can correctly apply the basic integration rules to elementary functions. |

### 5.7.1 Integrals Involving Inverse Trigonometric Functions

The derivatives of the six trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin(x)$ as the antiderivative of $1/\sqrt{1-x^2}$ rather than $-\arccos(x)$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you.

**Theorem 5.18**: Integrals Involving Inverse Trigonometric Functions

Let $u$ be a differentiable function of $x$, and let $a > 0$.

1. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\left(\frac{u}{a}\right) + C$
2. $\displaystyle\int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
3. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

**Example 5.35**: Evaluate the following:

1. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}dx$

2. $\displaystyle\int \frac{1}{2+9x^2}dx$

The integrals in that example are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

**Example 5.36**: Find $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}dx$.

**Example 5.37**: Find $\int \frac{x+2}{\sqrt{4-x^2}}dx.$

## Practice Exercises

Evaluate the following definite and indefinite integrals

**58.** $\int \frac{5}{1+25x^2}\,dx$

**59.** $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

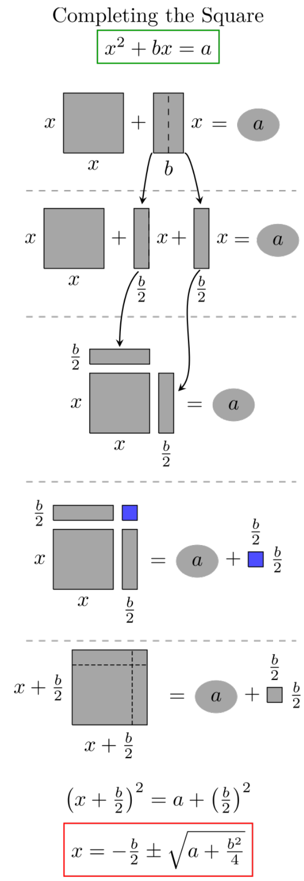
**60.** $\int \frac{e^x}{1+e^{2x}}\,dx$

**61.** $\displaystyle\int \frac{x+3}{\sqrt{1-x^2}}\,dx$

**62.** $\displaystyle\int \frac{1}{x\sqrt{x^2-16}}\,dx$

**63.** $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

## 5.7.2 Completing the Square




Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + C \end{aligned}$$

I would like to take this moment to note that you may be unfamiliar with this when it is written algebraically. You were probably taught to complete the square using an area model, which is fine, but calculus requires a strong understanding of algebraic manipulation. Furthermore, an area model does not generalize well to cubics or quartics (in my opinion); this means that while you might intuitively understand where the square has been "completed", the skill needs to be understand algebraically.

**Example 5.38**: Find $\displaystyle\int \frac{1}{x^2 - 4x + 7} dx$.

If the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1)] \end{aligned}$$

To complete the square when the coefficient of $x^2$ is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] \\ &= \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 \end{aligned}$$

### 5.7.3 Review of Basic Integration Rules

You have now completed the introduction of the basic integration rules. To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory

## Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du$
   $= \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{1}{u}du = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \dfrac{1}{\ln a}a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du$
    $= -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{1}{\sqrt{a^2 - u^2}}du = \arcsin\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{1}{a^2 + u^2}du = \dfrac{1}{a}\arctan\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{1}{u\sqrt{u^2 - a^2}}du$
    $= \dfrac{1}{a}\operatorname{arcsec}\left(\dfrac{|u|}{a}\right) + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate any elementary function. For integration, this is far from true. The integration rules listed above are primarily those that were happened on when developing differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper $du$ corresponding to the $u$ in the formula. The point is that you need to work on more integration techniques, which are in Chapter 8.

However, we will not explore them in this class. AP Calculus BC, also known as *Calculus 2*, is where you explore more integration techniques. Specifically, they cover rules like *integration by parts* which is used to "undo" the product rule.