Chapter 7 Applications of Integration

Student Notes

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Contents

7.1 Area of a Region Between Two Curves

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Find the area of a region between two curves using integration	I can set up and evaluate an integral to find the area between two curves by identifying which function is on top and choosing the correct limits.
Find the area of a region between intersecting curves using integration	I can find where two curves intersect and use those points to set up and evaluate the integral(s) needed to find the total area.
Describe integration as an accumulation process	I can explain how a definite integral represents accumulation and describe it as adding up small pieces to find a total.

7.1.1 Area of a Region Between Two Curves

With a few modifications you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 7.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 7.2.

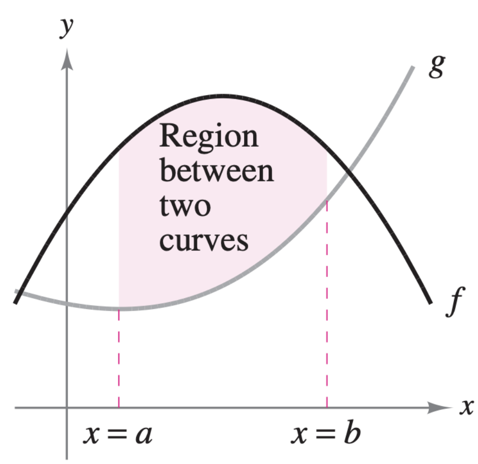


Figure 7.1 Area between two curves.

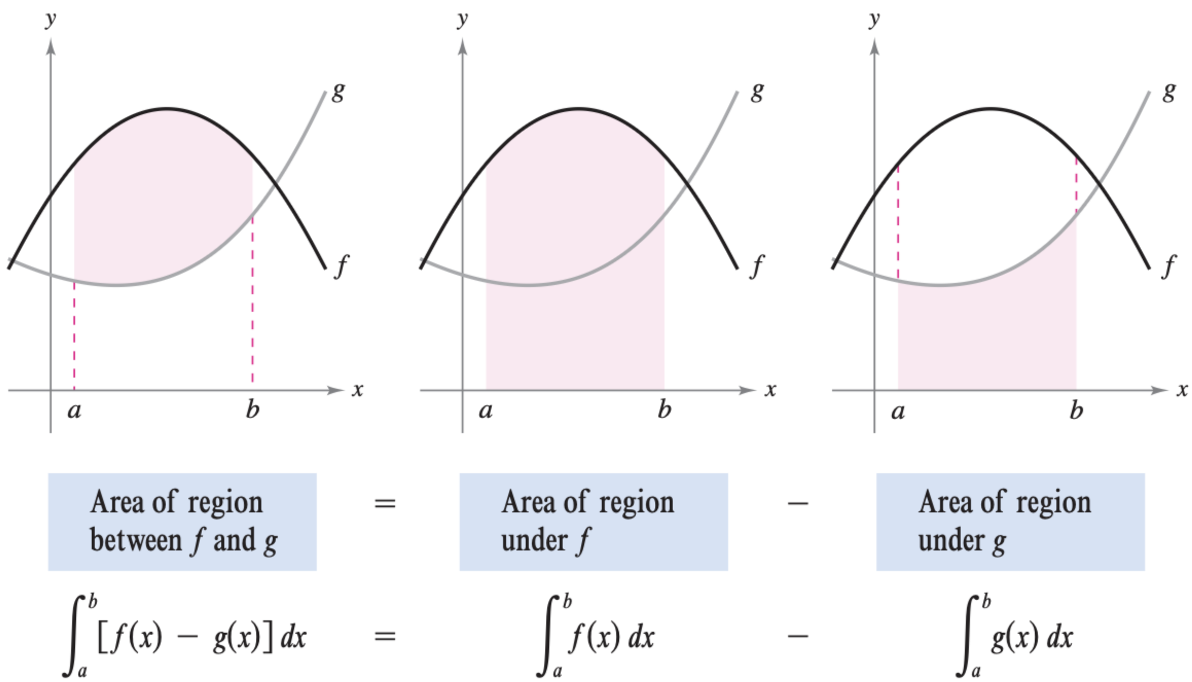


Figure 7.2 Area between two curves.

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith interval.
The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x$$

By adding the areas of the n rectangles and taking the limit as $||\Delta|| \to 0\,(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x$$

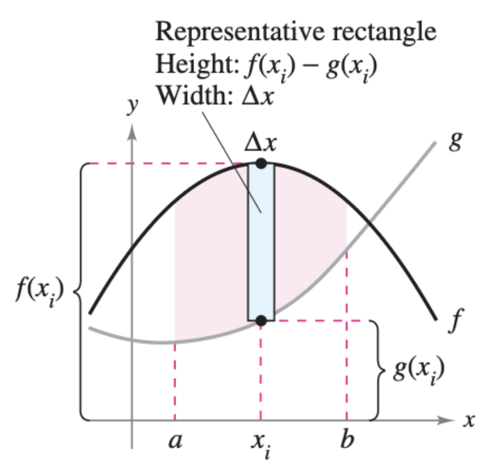


Figure 7.3

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So the area of the given region is

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x \\ &= \int_a^b [f(x) - g(x)]\,dx\end{aligned}$$

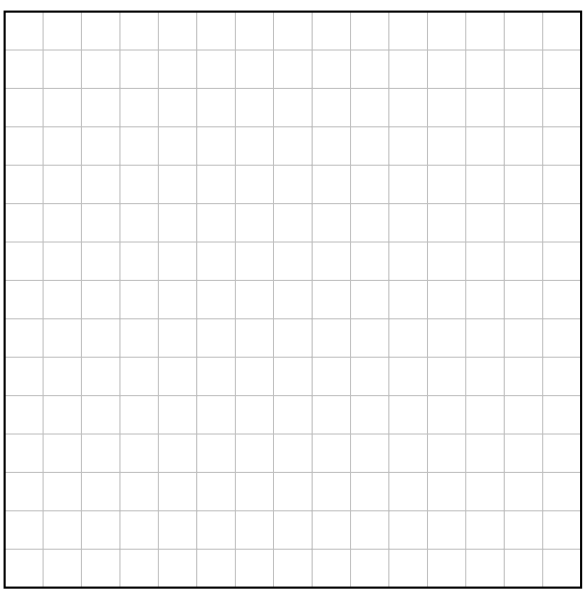
Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Note: The graphs do not need to be above the x-axis. The formula applies as long as $f(x)$ is the "top" curve and $g(x)$ is the "bottom" curve.

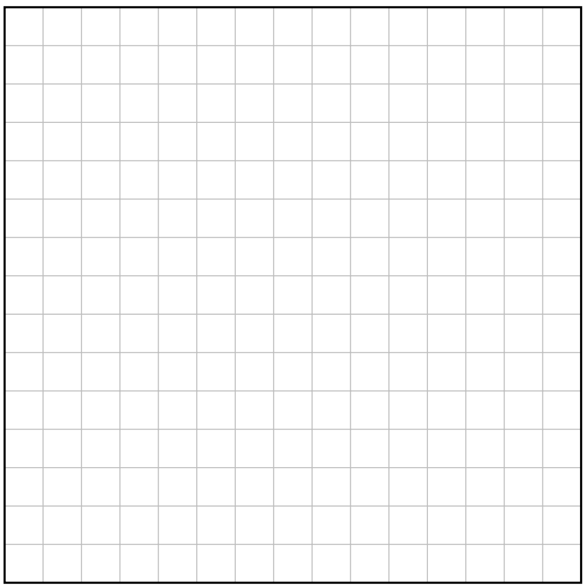
Example 7.1: Find the area of the region bounded by the graphs of $y = x^2 + 2, y = -x, x = 0$, and $x = 1$.



7.1.2 Area of a Region Between Intersecting Curves

In the previous example, the boundaries a and b were given to us explicitly. A more common problem requires you to find the area of a region bounded by two *intersecting* graphs. In these cases, you must calculate the intersection points to find your limits of integration a and b.

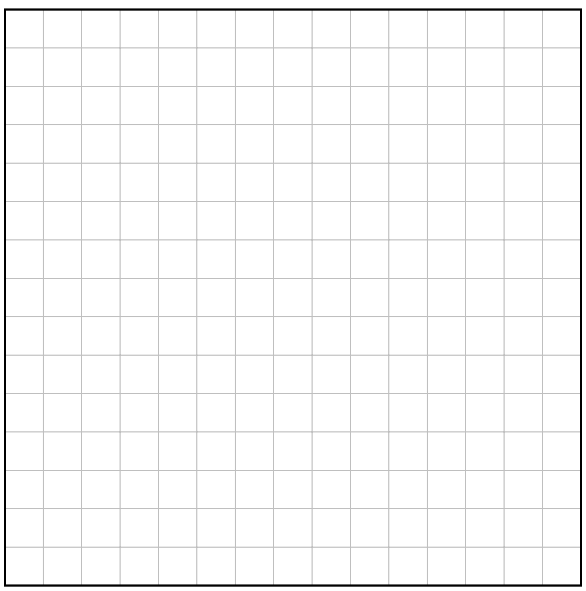
Example 7.2: Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



Example 7.3: The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions.

If two curves intersect at more than two points, you must find *all* points of intersection. You then need to check which curve is on top in each specific interval and write separate integrals.

Example 7.4: Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.



NOTICE: If you simply integrate from the leftmost intersection to the rightmost intersection without splitting the integral where the curves cross, you will subtract area instead of adding it, resulting in an incorrect answer!

Integrating with Respect to y (Horizontal Rectangles)

Sometimes, finding the top and bottom curves is overly complicated because the boundary changes. If the graph of a function of y is a boundary, it is often easier to use horizontal rectangles (with a height of Δy).

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x} \, dx$$

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y} \, dy$$

Example 7.5: Find the area of a region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Practice Exercises

1. Find the area of the region bounded by the graphs of $y = e^x$, $y = x$, $x = 0$, and $x = 2$.

2. Find the area of the region completely bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2 - 6$.

3. Let R be the region enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$. Find the area of R.

4. Find the area of the region in the first quadrant bounded by the graphs of $y = \sin(x)$, $y = \cos(x)$, and the y-axis.

5. Let R be the region bounded by the graphs of $f(x) = x^3 - 3x^2 + 2x$ and $g(x) = 0$. Find the total area of the region R.

6. Consider the region bounded by the graphs of $x = y^2 - 4$ and $x = y + 2$.

a) Write an integral expression with respect to x that represents the area of the region. Do not evaluate the integral.

b) Write an integral expression with respect to y that represents the area of the region.

c) Evaluate the integral from part (b) to find the area of the region.

7.1.3 Integration as an Accumulation Process

In this section, the integration formula for area was developed using a rectangle as the *representative element*. Throughout this chapter, we will take precalculus formulas, build a representative element, and sum them up using integration.

Known precalculus formula $\Longrightarrow$ Representative element $\Longrightarrow$ New integration formula

For area between two curves, the accumulation process looked like this:

$A = (\text{height})(\text{width})$ $\Longrightarrow$ $\Delta A = [f(x) - g(x)]\Delta x$ $\Longrightarrow$ $A = \int_a^b [f(x) - g(x)]\, dx$

Example 7.6: Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x-axis. Describe the integration as an accumulation process. (Check out the visual on Desmos!).

Conceptual Check

Q1. The Geometry of Accumulation
In the integral $\int_a^b [f(x) - g(x)]\, dx$, we are accumulating the areas of infinitely many representative rectangles. Geometrically, what specific parts of a single rectangle do the expressions $[f(x) - g(x)]$ and dx represent?

Q2. Area Below the x-axis
Even if two curves are completely below the x-axis, evaluating $\int_a^b [\text{top}-\text{bottom}]\,dx$ always results in a positive physical area. Algebraically, why does this specific subtraction guarantee a positive number for the height of your rectangles?

Q3. Crossing Curves
Suppose $f(x)$ and $g(x)$ intersect at $x = 0$, trading places as the "top" curve. If you evaluate the single integral $\int_{-2}^{2} [f(x) - g(x)]\,dx$ without splitting it at the intersection, why will your answer fail to represent the total area? What does this single integral actually calculate?

Q4. Choosing dy over dx
Sometimes it is much more efficient to integrate with respect to y instead of x. Geometrically, what happens to your vertical representative rectangles (dx) that forces you to write multiple integrals, and how do horizontal rectangles (dy) fix this?

Q5. Right vs. Left
For vertical rectangles (dx), we find the height using [Top − Bottom]. For horizontal rectangles (dy), the integrand is [Right − Left]. Explain why the "right" curve must be the first term in the subtraction.

7.2 Volume: The Disk Method

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Find the volume of a solid of revolution using the disk method	I can set up and evaluate an integral using the disk method by writing the radius in terms of the variable and choosing the correct limits.
Find the volume of a solid of revolution using the washer method	I can set up and evaluate an integral using the washer method by identifying the outer and inner radii and using the correct limits.
Find the volume of a solid with known cross sections	I can write an expression for the area of each cross section and use it to set up and evaluate an integral for the volume.

7.2.1 The Disk Method

In Chapter 4 we mentioned that area is only one of the *many* applications of the definite integral. Another important application is its use in finding the volume of a three-dimensional solid. In this section you will study a particular type of three-dimensional solid–one whose cross sections are similar. Solids of revolution are used commonly in engineering and manufacturing. Some examples are axles, funnels, pills, bottles, and pistons!

If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution**, and the line is called the **axis of revolution**. The simplest such solid is a right circular cylinder or **disk** which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in the figure on the right.

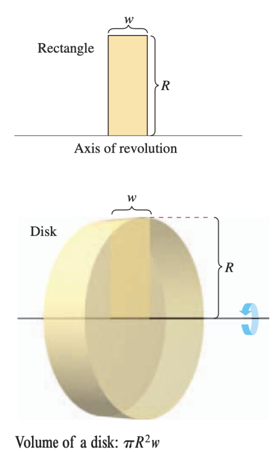


Figure 7.4

The volume of such a disk is

$$\begin{aligned}\text{Volume of disk} &= (\text{area of disk})(\text{width of disk})\\ &= \pi R^2 w\end{aligned}$$

where R is the radius of the disk and w is the width.

To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in the figure below about the indicated axis. To determine the volume of this solid, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x$$

Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\begin{aligned} \text{Volume of solid} &\approx \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x \\ &= \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x \end{aligned}$$

This approximation appears to become better and better as $||\Delta|| \to 0 \, (n \to \infty)$.

Definition of Volume of a Solid

$$\text{Volume of a solid} = \lim_{||\Delta|| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 \, dx$$

Volume of disk $V = \pi R^2 w$	$\Longrightarrow$	$\Delta V = \pi [R(x_i)]^2 \Delta x$	$\Longrightarrow$	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$

The Disk Method

To find the volume of a solid of revolution with the **disk method**, use one of the following:

Horizontal Axis of Revolution

$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 \, dx$$

Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_a^b [R(y)]^2 \, dy$$

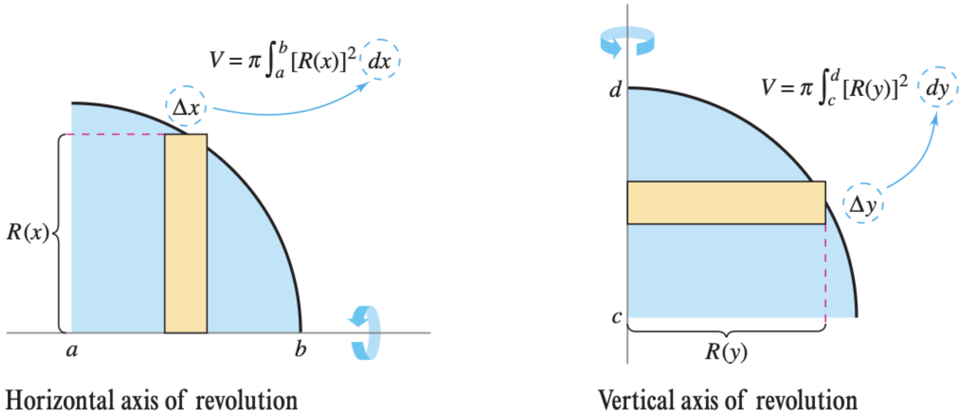


Figure 7.5 Determine the variable of integration by drawing your representative rectangle *perpendicular* to the axis of revolution.

Example 7.7: Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin(x)}$ and the x-axis $(0 \leq x \leq \pi)$ about the x-axis.

Example 7.8: Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the horizontal line $y = 1$.

Practice Exercises

7. Find the volume of the solid generated when the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the x-axis.

8. Find the volume of the solid generated when the region bounded by the graphs of $y = x^3$, $x = 0$, and $y = 8$ is revolved about the y-axis.

9. Find the volume of the solid generated when the region bounded by the graphs of $y = 2 - x^2$ and $y = 1$ is revolved about the horizontal line $y = 1$.

10. Let R be the region enclosed by the graph of $f(x) = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = e$. Which of the following expressions represents the volume of the solid generated when R is revolved about the x-axis?

(A) $\pi \int_1^e \frac{1}{x}\, dx$

(B) $\pi \int_1^e \frac{1}{x^2}\, dx$

(C) $2\pi \int_1^e \frac{1}{x^2}\, dx$

(D) $\pi \int_1^e \ln(x)\, dx$

11. Let R be the region in the first quadrant bounded by the graph of $y = 4 - x^2$ and the coordinate axes.

a) Find the area of R.

b) Find the volume of the solid generated when R is revolved about the x-axis.

c) Find the volume of the solid generated when R is revolved about the y-axis.

7.2.2 The Washer Method

When a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$ is revolved about an axis, the integral involving the inner radius represents the volume of the *hole* and is subtracted from the integral involving the outer radius.

The Washer Method

The volume of a solid using the washer method is given by:

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right)\, dx$$

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer**. If r and R are the inner and outer radii of the washer and w is the width, the volume is given by:

$$\text{Volume of washer} = \pi(R^2 - r^2)w$$

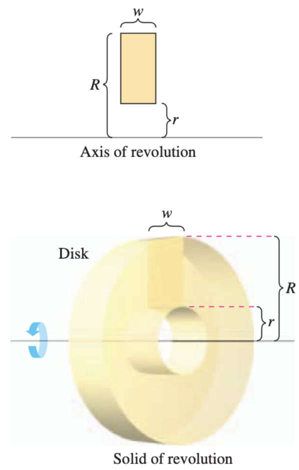


Figure 7.6 Washer Method

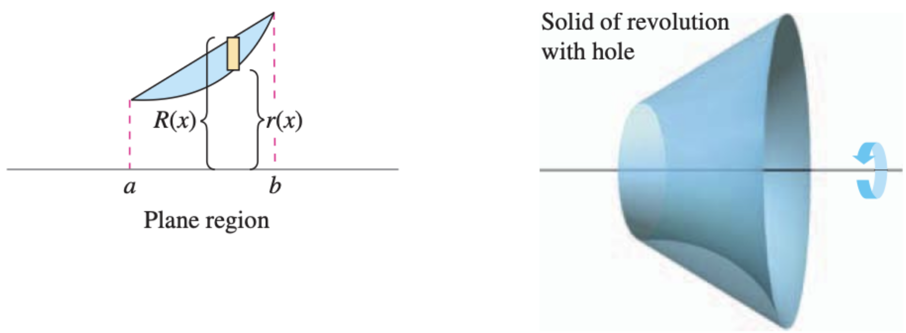
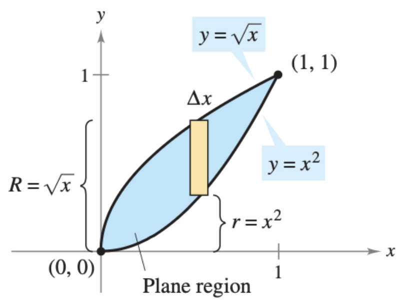
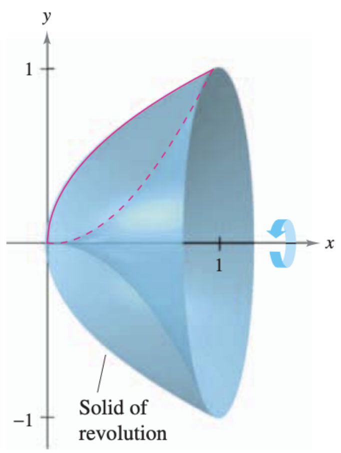


Figure 7.7

Example 7.9: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



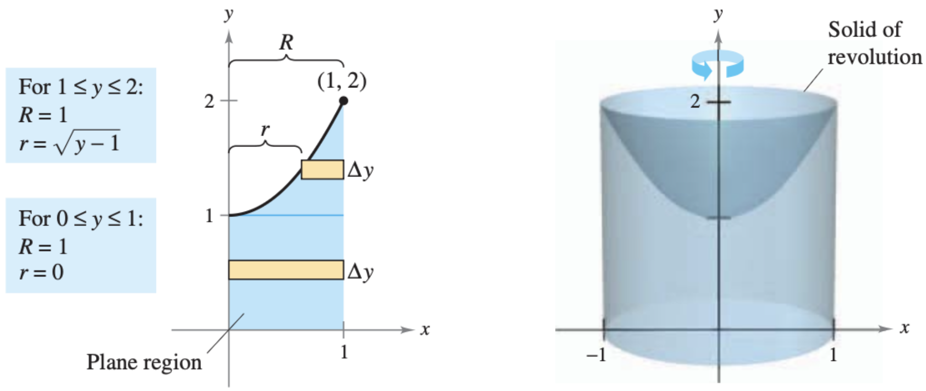


Solid of revolution

Figure 7.8

In each example so far, the axis of revolution has been *horizontal* and you have integrated with respect to x. In the next example, the axis of revolution is *vertical* and you integrate with respect to y. In this example, you need two separate integrals to compute the volume.

Example 7.10: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1, y = 0, x = 0$, and $x = 1$ about the y-axis.



y
R
(1, 2)
For $1 \le y \le 2$:
$R = 1$
$r = \sqrt{y-1}$
2
r
Δy
1
For $0 \le y \le 1$:
$R = 1$
$r = 0$
Δy
x
1
Plane region
y
Solid of
revolution
2
x
−1
1

Figure 7.9

Example 7.11: A manufacturer drills a hole through the center of a metal sphere of radius 5 inches. The hole has a radius of 3 inches. What is the volume of the resulting metal ring?

Practice Exercises

12. Let R be the region bounded by the graphs of $y = x^2$ and $y = 2x$. Find the volume of the solid generated when R is revolved about the x-axis.

13. Let R be the region bounded by the graph of $x = y^2$ and the vertical line $x = 4$. Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the y-axis.

14. Let R be the region enclosed by the graphs of $y = \sqrt{x}$ and $y = x^2$. Which of the following integrals represents the volume of the solid generated when R is revolved about the horizontal line $y = 2$?

(A) $\pi \int_0^1 \left((2 - x^2)^2 - (2 - \sqrt{x})^2\right)\, dx$

(B) $\pi \int_0^1 \left((2 - \sqrt{x})^2 - (2 - x^2)^2\right)\, dx$

(C) $\pi \int_0^1 \left(\sqrt{x} - x^2\right)^2\, dx$

(D) $\pi \int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right)\, dx$

15. Let R be the region bounded by the graphs of $y = x^2$ and $y = 4$. Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the horizontal line $y = 5$.

16. Let R be the region bounded by the graph of $y = e^x$, the horizontal line $y = 1$, and the vertical line $x = 2$.

a) Find the area of R.

b) Find the volume of the solid generated when R is revolved about the x-axis.

c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the horizontal line $y = -1$.

Conceptual Check

Q6. Disk vs. Washer
Geometrically, what specific feature of a bounded region and its relationship to the axis of revolution forces you to use the washer method instead of the disk method?

Q7. The Classic Algebra Mistake
When setting up a washer method integral, a common mistake is to write the integrand as $\pi[R(x)-r(x)]^2$. Algebraically and geometrically, why is this incorrect, and why must it be written as $\pi([R(x)]^2 - [r(x)]^2)$ instead?

Q8. Choosing the Variable (dx vs. dy)
If you are revolving a region around a vertical axis (like the y-axis or $x = 3$), you must integrate with respect to y. Geometrically, why does a vertical axis of revolution force your representative rectangles to have a width of dy?

Q9. Shifted Axes of Revolution
Suppose you are revolving a region around the line $y = -2$ instead of the x-axis ($y = 0$). How do you use the "Top − Bottom" concept to correctly write the radius $R(x)$, and why does this result in adding 2 to your function?

Q10. The Geometry of the Integrand

Integration is an accumulation process. In the disk method formula $V = \int_a^b \pi[R(x)]^2\,dx$, what specific 3D geometric shape does the expression $\pi[R(x)]^2\,dx$ represent before the integral adds them all together?

7.2.3 Solids with Known Cross Sections

With the disk method, we found the volume of a solid having a circular cross section area ($A = \pi R^2$). This method can be generalized to solids of *any* shape, as long as you know a formula for the area of an arbitrary cross section (like squares, rectangles, triangles, or semicircles).

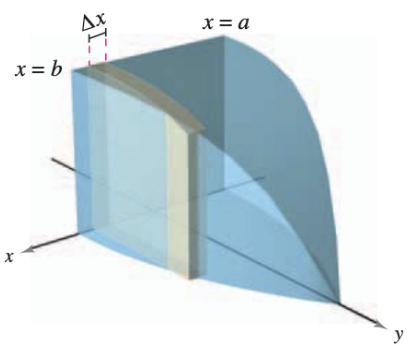
Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$\text{Volume} = \int_a^b A(x)\,dx \quad \text{Figure 7.14(a)}$$

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$\text{Volume} = \int_c^d A(y)\,dy \quad \text{Figure 7.14(b)}$$



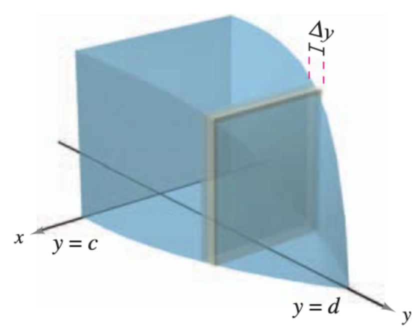
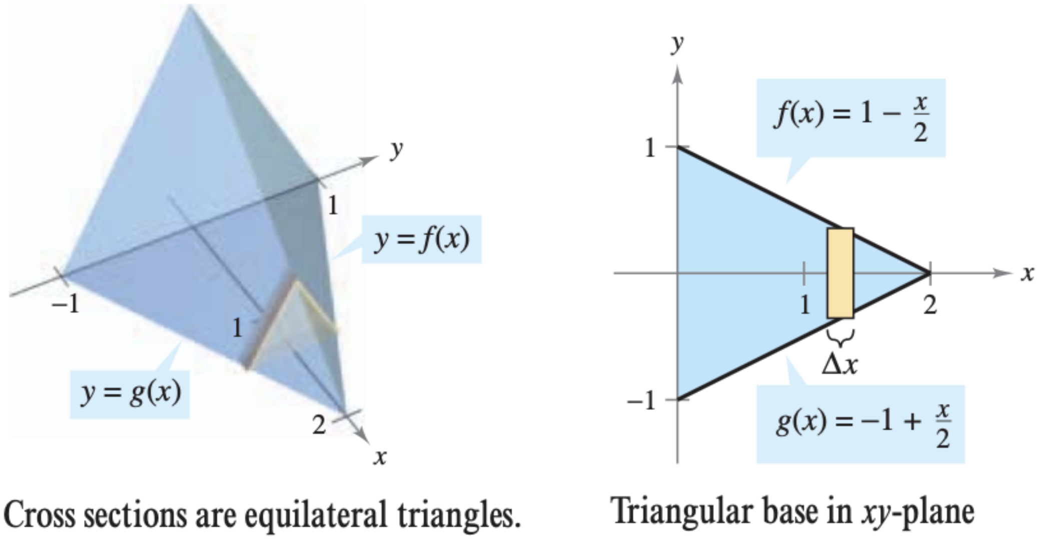


Figure 7.10 Cross Sections Perpendicular to a Particular Axis

Example 7.12: Find the volume of the solid shown. The base of the solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \quad g(x) = -1 + \frac{x}{2}, \quad \text{and} \quad x = 0$$

The cross sections perpendicular to the x-axis are equilateral triangles.



(a) First view (b) Second view

Figure 7.11 The base region and the resulting 3D solid.

Example 7.13: Prove that the volume of a pyramid with a square base is $V = \frac{1}{3}hB$, where h is the height of the pyramid and B is the area of the base.

Practice Exercises

17. Let R be the region bounded by the graph of $y = \sqrt{x}$, the x-axis, and the vertical line $x = 9$. The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

18. Let R be the region bounded by the graphs of $y = 2 - x^2$ and $y = 0$. The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of the solid.

19. Let R be the region bounded by the graphs of $y = e^x$, $y = 1$, and $x = 3$. The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are rectangles whose heights are twice the lengths of their bases in region R. Write, but do not evaluate, an integral expression for the volume of the solid.

20. Let R be the region bounded by the graph of $y = \ln(x)$, the x-axis, and the vertical line $x = e$. The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Which of the following integrals represents the volume of the solid?

(A) $\int_1^e \ln(x)\, dx$

(B) $\int_1^e (\ln(x))^2\, dx$

(C) $\pi \int_1^e (\ln(x))^2\, dx$

(D) $\pi \int_1^e \left(\frac{\ln(x)}{2}\right)^2 dx$

21. Let R be the region enclosed by the graphs of $x = y^2$ and $x = 4$. The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are equilateral triangles. Write, but do not evaluate, an integral expression for the volume of the solid.

22. Let R be the region bounded by the graphs of $y = \sin(x)$ and the x-axis for $0 \le x \le \pi$.

a) Find the area of R.

b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

c) The region R is the base of another solid. For this solid, the cross sections perpendicular to the x-axis are semicircles with their diameters across the base. Find the volume of the solid.

7.3 Volume: The Shell Method

Lesson Objectives & Success Criteria

Key Topics & Formulas	**Success Criteria**
Find the volume of a solid of revolution using the shell method	I can set up and evaluate an integral using the shell method by identifying the radius and height and choosing the correct limits.
Compare the uses of the disk method and the shell method	I can decide which method (disk/washer or shell) is easier for a problem and explain my choice based on how the region is oriented.

7.3.1 The Shell Method

In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the **shell method** because it uses cylindrical shells.

To begin, consider a representative rectangle as shown below:

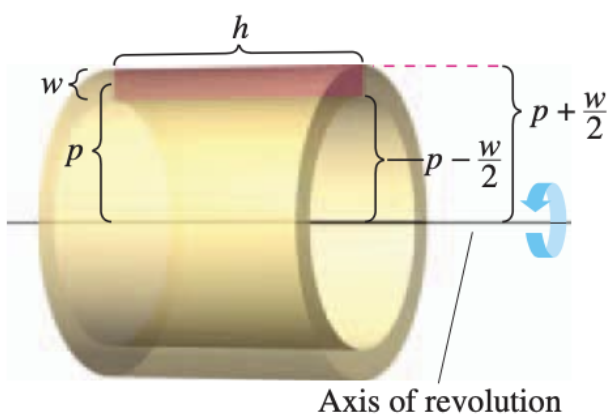


Figure 7.12

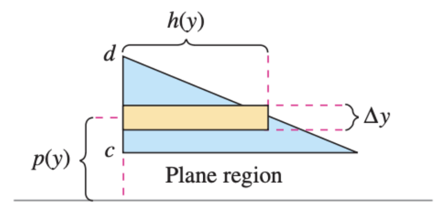
where w is the width of the rectangle, h is the height of the rectangle, and p is the distance between the axis of revolution and the *center* of the rectangle. When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness w. To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because p is the average radius of the shell, you know the outer radius is $p + (w/2)$ and the inner radius is $p - (w/2)$.

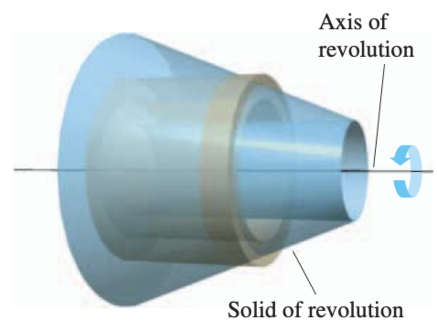
$$p + \frac{w}{2} \qquad \text{Outer radius}$$
$$p - \frac{w}{2} \qquad \text{Inner radius}$$

So, the volume of the shell is

$$\begin{aligned}\text{Volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole}) \\ &= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h \\ &= 2\pi phw \\ &= 2\pi(\text{average radius})(\text{height})(\text{thickness})\end{aligned}$$

You can use this formula to find the volume of a solid of revolution. Assume that the plane region in the figure on the right is revolved about a line to form the indicated solid. If you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is





$$\Delta V = 2\pi[p(y)h(y)]\Delta y$$

You can approximate the volume of the solid by n such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\Delta y = 2\pi \sum_{i=1}^{n}[p(y_i)h(y_i)]\Delta y$$

This approximation appears to become better and better as $||\Delta|| \to 0$ ($n \to \infty$). So, the volume of the solid is

$$\begin{aligned}\text{Volume of solid} &= \lim_{||\Delta|| \to 0} 2\pi \sum_{i=1}^{n}[p(y_i)h(y_i)]\Delta y \\ &= 2\pi \int_c^d [p(y)h(y)]\, dy\end{aligned}$$

The Shell Method

To find the volume of a solid of revolution with the **shell method**, use one of the following, as shown in the figure below.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$

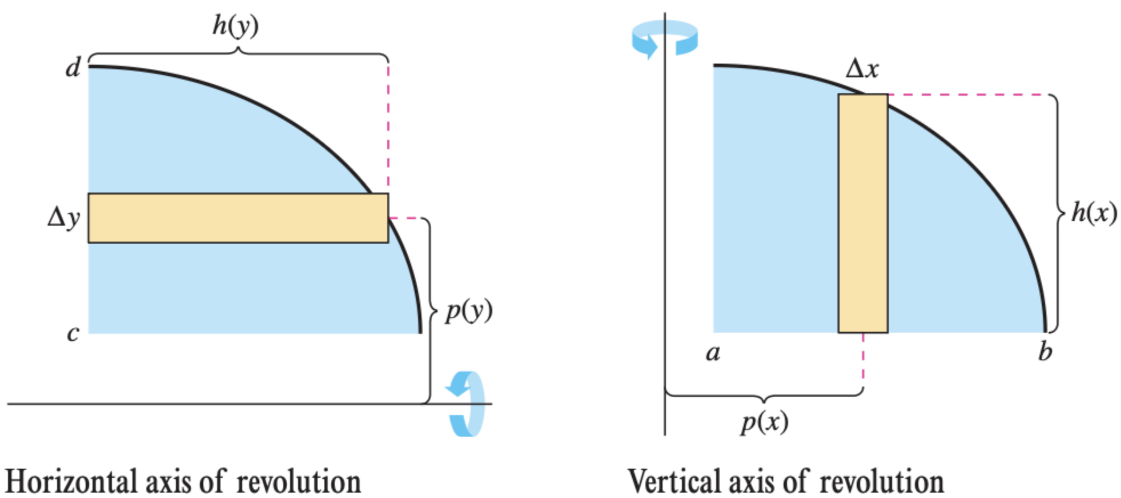
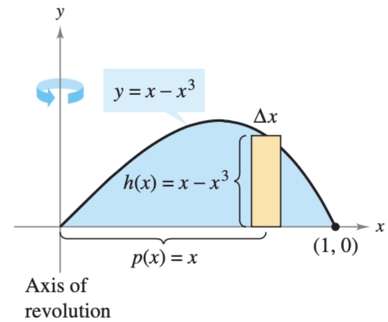
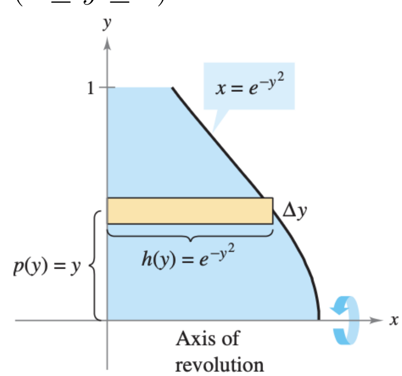


Figure 7.13

Example 7.14: Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis, $0 \le x \le 1$, about the y-axis.



Example 7.15: Find the volume of the solid of revolution formed by revolving the region bounded by the graph of $x = e^{-y^2}$ and the y-axis ($0 \le y \le 1$) about the x-axis.



7.3.2 Comparison of Disk and Shell Methods

The disk and shell methods can be distinguished as follows:

- **Disk/Washer Method:** The representative rectangle is always *perpendicular* to the axis of revolution.
- **Shell Method:** The representative rectangle is always *parallel* to the axis of revolution.

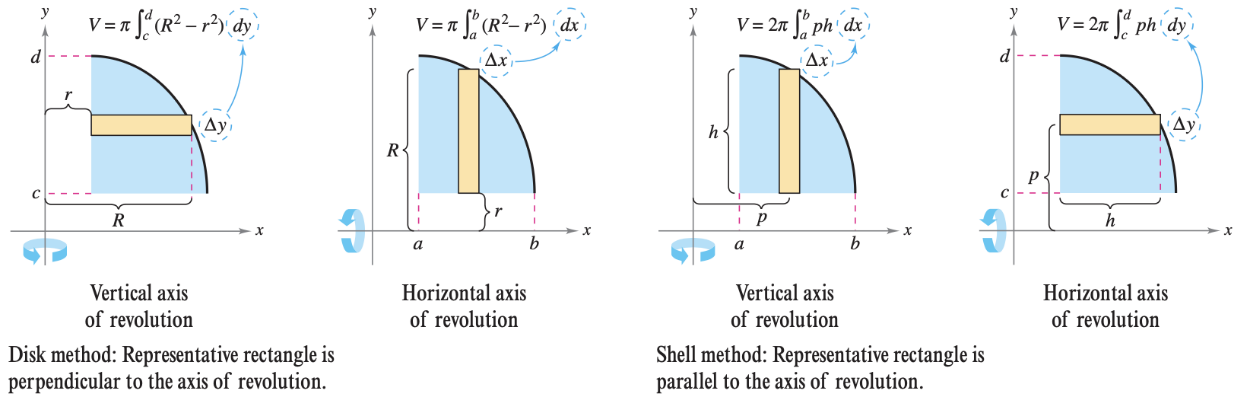


Figure 7.14 Perpendicular vs. Parallel representative rectangles.

Depending on the situation, it may be much more convenient to use one method over the other.

Example 7.16: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

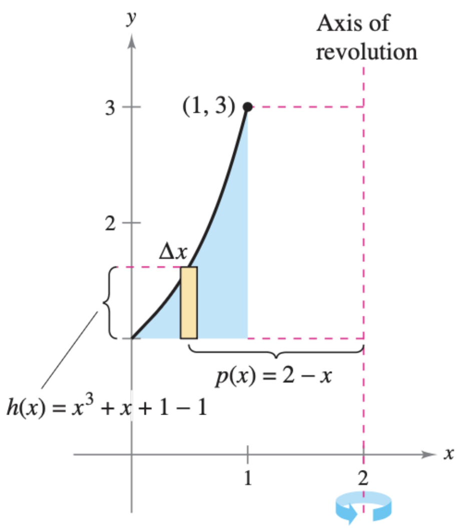
Note: This is the exact same problem as Example 7.10. Let's set it up both ways to see which is more efficient!

Method 1: The Washer Method

Method 2: The Shell Method

In some cases, solving for x in terms of y is very difficult (or even impossible). In such cases, you are forced to use a vertical rectangle (of width Δx), making x the variable of integration. Your choice of method is then determined entirely by whether the axis of revolution is vertical or horizontal.

Example 7.17: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1, y = 1$, and $x = 1$ about the line $x = 2$, as shown in the figure below.



Practice Exercises

23. Let R be the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$. Use the shell method to find the volume of the solid generated when R is revolved about the y-axis.

24. Let R be the region bounded by the graphs of $x = y^2$ and $x = 4$. Use the shell method to find the volume of the solid generated when R is revolved about the x-axis.

25. Let R be the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$. Write, but do not evaluate, an integral expression using the shell method for the volume of the solid generated when R is revolved about the vertical line $x = 6$.

26. Let R be the region in the first quadrant bounded by the graph of $y = \sin(x^2)$, the x-axis, and the vertical line $x = \sqrt{\pi}$. Which of the following integrals represents the volume of the solid generated when R is revolved about the y-axis?

(A) $\pi \int_0^{\sqrt{\pi}} (\sin(x^2))^2 \, dx$

(B) $2\pi \int_0^{\sqrt{\pi}} x \sin(x^2) \, dx$

(C) $2\pi \int_0^{\pi} y \sin(y^2) \, dy$

(D) $\pi \int_0^{\sqrt{\pi}} x^2 \sin(x^2) \, dx$

27. Let R be the region bounded by the graphs of $y = x^3 - x^4$ and the x-axis.

a) Briefly explain why using the disk/washer method to find the volume of the solid generated when R is revolved about the y-axis would be difficult.

b) Write an integral expression using the shell method that represents the volume of the solid generated when R is revolved about the y-axis.

c) Evaluate the integral from part (b) to find the volume of the solid.